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Algebraic Geometry II

5. Exercise sheet

Exercise 1:
Let $X \to S$ be a morphism of schemes. We define a functor, the "tangent bundle" $\mathcal{T}_{X/S}$ of X/S, by sending an affine scheme $\mathrm{Spec}(A)$ over S to the set $X(A[\varepsilon])$. Prove that $\mathcal{T}_{X/S}$ is representable by the relative spectrum $\mathrm{Spec}(\mathrm{Sym}^\bullet(\Omega^1_{X/S})) \to X$.
Hint: Use the natural morphism $\mathcal{T}_{X/S} \to X$ induced by $\epsilon \mapsto 0$ to reduce to the case that X and S are affine.

Exercise 2:
Let k be an algebraically closed field of characteristic $\neq 2, 3$. For $a, b \in k$ with $4a^3 + 27b^2 \in k^\times$ consider the elliptic curve

$$E := V(ZY^2 - X^3 - aXZ^2 - bZ^3) \subset \mathbb{P}^2_k.$$

Let $y = \frac{Y}{Z}$ and $x = \frac{X}{Z}$ inside the function field $K(E)$. Let $\omega := \frac{dx}{2y} \in \Omega_{K(E)/k}$.

1. Show that ω extends to a section of $H^0(E, \Omega_{E/k})$.
2. Show that $\Omega_{E/k} = \omega\mathcal{O}_E$, i.e., that $\Omega_{E/k}$ is free of rank 1 generated by the global section ω.

Exercise 3: Let $f\colon X \to S$ a morphism locally of finite type. Assume that S is a Dedekind scheme (see Sheet 2, Exercise 3), and that X is reduced. Show that the following assertions are equivalent:

1. The map f is flat.
2. The map f is open.
3. The map f is universally open.
4. Every irreducible component of X dominates S, i.e., whenever η is the generic point of an irreducible component of X, then $f(\eta)$ is the generic point of S.

Exercise 4:
Let k'/k be a Galois extension of fields with Galois group $G = \mathrm{Aut}_{k\text{-Alg}}(k')$. A semi-linear G-action on a k'-vector space V' is a collection $(\varphi_g)_{g\in G}$ of isomorphisms on the underlying abelian groups $\varphi_g\colon V' \to V'$ satisfying $\varphi_g(\lambda \cdot v) = g(\lambda) \cdot \varphi_g(v)$ for all $\lambda \in k'$, $v \in V'$, and $\varphi_{gg'} = \varphi_g \circ \varphi_{g'}$ for all $g, g' \in G$.

1. Show that the pairs $(V', (\varphi_g)_{g\in G})$ of vector spaces equipped with a semi-linear G-action naturally forms a category, denoted $\mathrm{Vec}(k'/k)$.

2. For an object $\big(V', (\varphi_g)_{g\in G}\big)$ in $\mathrm{Vec}(k'/k)$ we denote by $(V')^G$ its fixed points, i.e., the subset of all $v \in V'$ such that $\varphi_g(v) = v$ for all $g \in G$. Show that $(V')^G$ is a k-subvector space of V'.

3. Show that the functor $V \mapsto \big(k' \otimes_k V, (g \otimes \mathrm{id})_{g\in G}\big)$ induces an equivalence of categories

$$(k\text{-vector spaces}) \xrightarrow{\cong} \mathrm{Vec}(k'/k)$$

with quasi-inverse $\big(V', (\varphi_g)_{g\in G}\big) \mapsto (V')^G$.

Hint: For part 3. use the isomorphism of k'-algebras $k' \otimes_k k' \simeq \prod_{g\in G} k', a \otimes b \mapsto (a \cdot g(b))_{g\in G}$ and fpqc descent for quasi-coherent sheaves.