Prof. Dr. Timo Richarz
M. Sc. Can Yaylali

WS 2019/20

Algebraic Geometry II

4. Exercise sheet

Exercise 1:
Let k be an algebraically closed field of characteristic 0, and let $n \geq 2$ be an integer. We consider the plane curve

$$C = V(Y^2Z^{n-2} - X^n + Z^n) \subset \mathbb{P}^2_k.$$

This is a projective and integral curve over k.

1. Show that C is regular for $n \leq 3$, and that C is not regular for $n \geq 4$.
2. Compute the normalization of C if $n \geq 4$.

Hint: For 2. distinguish between the cases n odd and n even.

Exercise 2:
Let $f\colon X \to Y$ and $g\colon Y \to Z$ morphism of schemes. Assume that f is smooth of relative dimension d in a point $x \in X$, and that g is smooth of relative dimension e in $f(x) \in Y$. Show that $g \circ f$ is smooth of relative dimension $d + e$ in x.

Exercise 3: Let $A \to B \to C$ be morphisms of rings.

1. Show that there exists an exact sequence of C-modules

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

2. Assume that $B \to C$ is surjective with kernel $I := \ker(B \to C)$. Prove that there exists an exact sequence of C-modules

$$C \otimes_B I \cong I/I^2 \xrightarrow{\alpha} C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0,$$

where $\alpha(g) := 1 \otimes dg$ for $g \in I$.

Exercise 4:
Let $A \to B$ be a morphism of rings. Compute the Kähler differentials $\Omega_{B/A}$ for

1. $B = A[X]/(f(X))$ with $f(X) \in A[X]$ a polynomial.
2. $B = \mathbb{Z}[i]$, $A = \mathbb{Z}$.
3. $B = k[x, y]/(y^2 - x^3 - x)$, and $A = k$ a field not of characteristic 2.
4. $B = A[x, y]/(xy)$.

Hint: Use Exercise 3.2.