Prof. Dr. Timo Richarz
M. Sc. Can Yaylali **WS 2019/20**

Algebraic Geometry II

3. Exercise sheet

Exercise 1:
Let k be an algebraically closed field. Let $C \to \mathrm{Spec}(k)$ be a proper curve which is normal and connected. For each closed point $x \in C$ the local ring $\mathcal{O}_{C,x}$ is a discrete valuation ring, and we denote by $v_x\colon K(C)^\times \to \mathbb{Z} \cup \{\infty\}$ the associated valuation of its function field. For $C = \mathbb{P}^1_k$ and $f \in K(C)^\times$ show that

$$\sum_{x \in C_{\mathrm{cl}}} v_x(f) = 0,$$

where $C_{\mathrm{cl}} \subset C$ denotes the subset of closed points.

Exercise 2:
Let X be a scheme. Show that the following are equivalent:

1. The structure sheaf $\mathcal{O}_X$ is ample.
2. The scheme X is quasi-compact, and the open subsets $D(s)$ for $s \in \Gamma(X, \mathcal{O}_X)$ form a basis of the topology for X.
3. The canonical map $X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ is a quasi-compact open immersion.
4. There exists a quasi-compact open immersion $X \to Y$ where Y is an affine scheme.

A scheme which satisfies one of the equivalent conditions 1.-4. is called *quasi-affine.*

Exercise 3:
Let X be a scheme equipped with an ample line bundle $\mathcal{L}$.

1. Show that for all $m > 0$ and $s \in \Gamma(X, \mathcal{L}^{\otimes m})$ the open set $D(s) \subseteq X$ is quasi-affine.
2. Let $x_1, \dots, x_n \in X$ be points. Prove that there exists an $m > 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that $D(s)$ is affine and $x_1, \dots, x_n \in D(s)$.

Hint: For part 1 use Exercise 2. For part 2 define prime ideals $\mathfrak{p}_i$ in the ring $R := \oplus_{d \geq 0} \Gamma(X, \mathcal{L}^{\otimes d})$ by

$$f \in \mathfrak{p}_i \Leftrightarrow f(x_i) = 0.$$

Then use prime avoidance in the ring R to find $s' \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that $x_1, \dots, x_n \in D(s')$. Now use part 1 to reduce to the case where X is quasi-affine.

Exercise 4:
Let X be a scheme admitting an ample line bundle. Prove that X is separated.

Hint: Use Exercise 3 and the valuative criterion for separatedness.