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Algebraic Geometry II

2. Exercise sheet

Exercise 1:

Let k be an algebraically closed field, and let $X \to \mathrm{Spec}(k)$ be a proper scheme. Assume that X is connected and reduced. Show that $\Gamma(X, \mathcal{O}_X) = k$.

Hint: Every function $f \in \Gamma(X, \mathcal{O}_X)$ induces a morphism, also denoted $f: X \to \mathbb{A}^1_k$. Consider its composition with the inclusion $\mathbb{A}^1_k \subset \mathbb{P}^1_k$.

Exercise 2:

Let V be a valuation ring.

1. Show that V is a local, normal domain.
2. Show that V is a Bézout domain, i.e., that every finitely generated ideal is principal.
3. Show that the following are equivalent:

   The ring V is a discrete valuation ring or a field.

   The ring V is a principal ideal domain.

   The ring V is Noetherian.

Exercise 3:

Let S be a scheme, and let X, Y be schemes over S. Assume that $X \to S$ is proper, and that Y is a Dedekind scheme, i.e., Y is a Noetherian, normal, integral scheme of dimension ≤ 1. Let K be the function field of Y, and denote by $\iota: \mathrm{Spec}(K) \to Y$ the canonical map. Show that the map

$$\mathrm{Hom}_S(Y, X) \to \mathrm{Hom}_S(\mathrm{Spec}(K), X), \quad f \mapsto f \circ \iota,$$

is a bijection of sets. In other words, every S-morphism $\mathrm{Spec}(K) \to X$ extends uniquely to an S-morphism $Y \to X$.

Exercise 4:

Use the valuative criteria to show the following statements:

1. Let $0 \leq d \leq n$ integers. Show that the Grassmannian $\mathrm{Grass}_{n,d} \to \mathrm{Spec}(\mathbb{Z})$ defined in Algebraic Geometry I, Exercise 11.1 is proper. For example, this shows that the projective space $\mathbb{P}^n_{\mathbb{Z}} = \mathrm{Grass}_{n+1,n} \to \mathrm{Spec}(\mathbb{Z})$ is proper.
2. Let k be a field, and let $X = \mathbb{A}^1_k \sqcup_{\mathbb{A}^1_k \backslash \{0\}} \mathbb{A}^1_k$ be the affine line with double origin defined by gluing two copies of $\mathbb{A}^1_k$ along the open subsets $\mathbb{A}^1_k \backslash \{0\}$. Show that X is not separated.