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Algebraic Geometry II

1. Exercise sheet

Exercise 1:
Let $\mathcal{P}$ be one of the following properties of morphisms of schemes: "affine", "quasi-finite", "finite", "integral".

(i) Prove that property $\mathcal{P}$ is stable under composition, base change and local on the target.

(ii) Let S be a scheme, and let X, Y be S-schemes. Let $f\colon X \to Y$ be a morphism of S-schemes. Assume that $X \to S$ has property $\mathcal{P}$, and that $Y \to S$ is separated. Show that the morphism f has property $\mathcal{P}$.

Exercise 2:
Let X be a locally Noetherian normal scheme, and let $U \subset X$ be an open subset whose complement $X \backslash U$ has codimension ≥ 2 in X. Show that the restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is an isomorphism. In other words, every function $f \in \Gamma(U, \mathcal{O}_X)$ extends uniquely from U to X.

Hint: Use the following fact from commutative algebra: Let A be a Noetherian integral domain. Then A is normal if and only if $A = \cap_{\mathfrak{p}} A_{\mathfrak{p}}$ where the intersection is taken over all prime ideals $\mathfrak{p}$ in A of height 1 (i.e., $\mathfrak{p} \neq (0)$, and if $(0) \subset \mathfrak{p}' \subset \mathfrak{p}$ then either $\mathfrak{p}' = (0)$ or $\mathfrak{p}' = \mathfrak{p}$).

Exercise 3:
Let X be a scheme, and let U be an open subscheme. Denote by $j\colon U \to X$ the inclusion.

(i) Assume that U is affine and that X is separated. Show that j is an affine morphism.

(ii) Assume that X is Noetherian and that j is an affine morphism. Show that every irreducible component of $X \backslash U$ has codimension ≤ 1 in X.

Hint: For (ii), follow the hint in Görtz-Wedhorn, Exercise 12.18.

Exercise 4:
Let k be a field. Let $\mathbb{A}^1_k = \operatorname{Spec}(k[t])$ be the affine line over k. Let $\mathbb{A}^2_k = \operatorname{Spec}(k[x, y])$ be the affine plane over k. Consider the following two cases:

(i) Let $X := V(y^2 - x^3) \subset \mathbb{A}^2_k$ be the cuspidal curve, and $f\colon \mathbb{A}^1_k \to X$ be the morphism induced from $x \mapsto t^2$, $y \mapsto t^3$ on rings.

(ii) Let $X := V(y^2 - x^2(x + 1)) \subset \mathbb{A}^2_k$ be the nodal curve, and let $f\colon \mathbb{A}^1_k \to X$ be the morphism induced from $x \mapsto t^2 - 1$, $y \mapsto t(t^2 - 1)$ on rings.

Show in each case that f is the normalization of X. If k is algebraically closed and $x \in X(k)$, then show that the set $f^{-1}(x)$ is finite and determine its cardinality.