Prof. Dr. T. Richarz
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SS 2019

Algebraic Geometry I

11. Exercise sheet

Exercise 1 (4 Points):
Let $n \geq d \geq 0$. Prove that the functor sending a scheme X to the isomorphism classes of quotients $\mathcal{O}_X^n \twoheadrightarrow \mathcal{E}$ with $\mathcal{E}$ locally free of rank $n-d$ is representable by a scheme $\mathrm{Grass}_{n,d}$ over $\operatorname{Spec}\mathbb{Z}$, called the *Grassmannian*.

Exercise 2 (4 Points):
Let $n \geq 0$.

(i) Prove that the functor

$$X \mapsto \{a : \mathcal{O}_X^n \to \mathcal{O}_X^n \,|\, a \text{ is an isomorphism}\}$$

is representable by a scheme, called GL_n, over $\operatorname{Spec}\mathbb{Z}$.

(ii) Let S be a scheme and let $\mathcal{E}$ be a vector bundle of rank n on S. Prove that the functor

$$(f : X \to S) \mapsto \{a : f^*\mathcal{E} \to f^*\mathcal{E} \,|\, a \text{ is an isomorphism}\}$$

is representable by a scheme, called $\mathrm{Aut}(\mathcal{E})$, over S, which is locally on S isomorphic to $\mathrm{GL}_n \times S$.

Exercise 3 (4 Points):
Let X be a scheme.

(i) Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. Show that the canonical maps

$$\mathcal{F}/\ker(f) \to \mathrm{im}(f), \qquad \mathcal{G}/\mathrm{im}(f) \to \mathrm{coker}(f)$$

are isomorphisms. Further, show that $\ker(f)$, $\mathrm{im}(f)$, and $\mathrm{coker}(f)$ are quasi-coherent if $\mathcal{F}$ and $\mathcal{G}$ are.

(ii) Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. Show that if $\mathcal{F}'$ and $\mathcal{F}''$ are quasi-coherent, so if $\mathcal{F}$. If X is locally noetherian, show that $\mathcal{F}$ is coherent if and only if $\mathcal{F}'$ and $\mathcal{F}''$ are.

Exercise 4 (4 Points):
Show that the property "(locally) of finite type" of morphisms of schemes is stable under composition and base change.

To be handed in during the lecture on: Wednesday, July 10.