Prof. Dr. T. Richarz
Dr. U. Buchholtz

SS 2019

Algebraic Geometry I

10. Exercise sheet

Exercise 1 (4 Points):

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of ringed spaces.

i) Show that the functors $g_* \circ f_*$ and $(g \circ f)_*$ from $\mathcal{O}_X$-modules to $\mathcal{O}_Z$-modules are equal.

ii) Show that the functors $f^* \circ g^*$ and $(g \circ f)^*$ from $\mathcal{O}_Z$-modules to $\mathcal{O}_X$-modules are canonically isomorphic.

Exercise 2 (4 Points):

Let X be a ringed space, and let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules.

i) Show that the presheaf on X defined by

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$$

is a sheaf and has the structure of an $\mathcal{O}_X$-module. This is denoted $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

ii) Assume additionally that X is a scheme, and that $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent. Show that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent whenever $\mathcal{F}$ is of finite presentation.

Exercise 3 (4 Points):

i) Prove that there exists a unique morphism $\sigma_{m,n}\colon \mathbb{P}^m_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{P}^n_{\mathbb{Z}} \to \mathbb{P}^{mn+m+n}_{\mathbb{Z}}$, called the *Segre embedding*, which induces for every scheme T the map

$$(\mathcal{O}_T^{m+1} \overset{\alpha}{\twoheadrightarrow} \mathcal{L}, \mathcal{O}_T^{n+1} \overset{\beta}{\twoheadrightarrow} \mathcal{L}') \mapsto (\alpha \otimes \beta\colon \mathcal{O}_T^{mn+m+n+1} \cong \mathcal{O}_T^{m+1} \otimes_{\mathcal{O}_T} \mathcal{O}_T^{n+1} \twoheadrightarrow \mathcal{L} \otimes_{\mathcal{O}_T} \mathcal{L}')$$

on T-valued points.

ii) Let T be the spectrum of a local ring. Show that

$$\sigma_{m,n}((x_0, \ldots, x_m), (y_0, \ldots, y_n)) = (x_i y_j)_{i,j}$$

under the bijection from sheet 8, exercise 2.

Exercise 4 (4 Points):

Let R be a ring, $n \geq 1$. Compute the global sections $\Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d))$, $d \in \mathbb{Z}$.

To be handed in during the lecture on: Wednesday, July 3.