Prof. Dr. T. Richarz **SS 2019**
Dr. U. Buchholtz

Algebraic Geometry I

8. Exercise sheet

Exercise 1 (4 Points):

Let R be a ring, $n \geq 0$.

i) Show that the natural homomorphism $R \to \Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R})$ is an isomorphism.

ii) Show that the scheme $\mathbb{P}^n_R$ is affine if and only if $n = 0$. In this case, $\mathbb{P}^0_R = \mathrm{Spec}(R)$.

Exercise 2 (4 Points):

i) Let R be a local ring. Show that the set $\mathbb{P}^n_{\mathbb{Z}}(R)$ is in natural bijection to the set of tuples $(x_0, \ldots, x_n)$ with $x_i \in R$ and some $x_j \in R^\times$ modulo the equivalence relation

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) \Leftrightarrow \exists\, \alpha \in R^\times : x_i = \alpha \cdot y_i \ \forall i.$$

Hint: Show that a morphism $\mathrm{Spec}(R) \to X$ to any scheme factors as $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ where $x \in X$ is the image of the closed point in $\mathrm{Spec}(R)$. Deduce that a morphism $\mathrm{Spec}(R) \to \mathbb{P}^n_{\mathbb{Z}}$ factors through some $D_+(X_j)$.

ii) Let $n, m \geq 0$ be two integers. Show that the schemes $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{P}^m_{\mathbb{Z}}$ and $\mathbb{P}^{n+m}_{\mathbb{Z}}$ are isomorphic if and only if $n = 0$ or $m = 0$.

Hint: Count k-valued points for k a finite field.

Exercise 3 (4 Points):

Let R be a ring, $n \geq 0$, and let $A = (a_{i,j})_{i,j} \in \mathrm{GL}_{n+1}(R)$ be an invertible $(n+1) \times (n+1)$ matrix with entries in R. Let $\mathbb{A}^{n+1}_R = \mathrm{Spec}(R[X_0, \ldots, X_n])$ with open subscheme $\mathbb{A}^{n+1}_R \backslash \{0\}$. We consider the morphism of R-schemes $\mathbb{A}^{n+1}_R \backslash \{0\} \to \mathbb{P}^n_R$ which on coordinates is given by

$$(x_0, \ldots, x_n) \mapsto (x_0 : \ldots : x_n).$$

i) The ring isomorphism $R[X_0, \ldots, X_n] \to R[X_0 \ldots, X_n]$ given by

$$X_i \mapsto \sum_j a_{i,j} X_j$$

induces an isomorphism $\mathbb{A}^{n+1}_R \to \mathbb{A}^{n+1}_R$ of R-schemes. Show that A restricts to an automorphism f_A of $\mathbb{A}^{n+1}_R \backslash \{0\}$.

ii) Show that there exists a unique automorphism f_A of $\mathbb{P}^n_R$ which fits into a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^{n+1}_R \backslash \{0\} & \longrightarrow & \mathbb{A}^{n+1}_R \backslash \{0\} \\ \downarrow & & \downarrow \\ \mathbb{P}^n_R & \longrightarrow & \mathbb{P}^n_R \end{array}$$

In this way we obtain a group homomorphism from $\mathrm{GL}_{n+1}(R)$ to the group $\mathrm{Aut}_R(\mathbb{P}^n_R)$ of automorphisms of the R-scheme $\mathbb{P}^n_R$.

iii) Now let $R = k$ be an algebraically closed field of characteristic $\neq 2$. Let $F \in k[X_0, \ldots, X_n]$ be homogenous of degree 2, $F \neq 0$. We call $V_+(F) \subset \mathbb{P}^n_k$ a *q*uadric. Which of the following quadrics in $\mathbb{P}^2_k$ are isomorphic as k-schemes?

$$V_+(X_0^2 + X_1^2), \;\; V_+(X_0^2 + X_1^2 + X_2^2), \;\; V_+(X_0^2 - X_1 X_2).$$

iv)* Show that quadrics over algebraically closed fields of characteristic $\neq 2$ are classified by their rank.

Hint. Remember from linear algebra that a quadratic form over a field of characteristic $\neq 2$ is the same as a symmetric bilinear form.

Exercise 4 (4 Points):

Let k be an algebraically closed field. Describe the fibers in all points of the following morphisms $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ corresponding in each case to the canonical morphism $A \to B$. Which fibers are irreducible or reduced?

i) $\mathrm{Spec}(k[T,U]/(TU - 1)) \to \mathrm{Spec}(k[T])$
ii) $\mathrm{Spec}(k[T,U]/(T^2 - U^2)) \to \mathrm{Spec}(k[T])$
iii) $\mathrm{Spec}(k[T,U]/(TU)) \to \mathrm{Spec}(k[T])$
iv) $\mathrm{Spec}(k[T,U,V]/(V^2 - TU)) \to \mathrm{Spec}(k[T])$

To be handed in during the lecture on: Wednesday, June 19.