Prof. Dr. T. Richarz **SS 2019**
Dr. U. Buchholtz

Algebraic Geometry I

7. Exercise sheet

Exercise 1 (4 Points):
Let $\mathcal{C}$ be a category. For $X \in \mathcal{C}$ let $h_X := \mathrm{Hom}_{\mathcal{C}}(-, X) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ be the associated functor. Let $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ be an arbitrary functor.
i) Prove the Yoneda lemma, i.e., that the map

$$\mathrm{Hom}(h_X, F) \to F(X),\ \phi \mapsto \phi(\mathrm{Id}_X)$$

is a bijection, natural in X and F.
ii) Let S be a scheme, and let $\mathcal{C} = (\mathrm{Sch}/S)$ be the category of schemes over S. Let $\mathcal{D} = (\mathrm{AffSch}/S)$ be the full subcategory consisting oyts $X/S \in \mathcal{C}$ with X affine. For any two S-schemes X, Y, show that there are bijections

$$\mathrm{Hom}_S(X, Y) \cong \mathrm{Hom}(h_X, h_Y) \cong \mathrm{Hom}(h_{X|\mathcal{D}}, h_{Y|\mathcal{D}})$$

where $F_{|\mathcal{D}}$ denotes the restriction of a functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ to $\mathcal{D}^{\mathrm{op}}$.

Exercise 2 (4 Points):
Let $(X, \mathcal{O}_X)$ be a locally ringed space.
i) Let $U \subseteq X$ be an open and closed subset. Show that there exists a unique section $e_U \in \Gamma(X, \mathcal{O}_X)$ such that $e_{U|V} = 1$ for all open subsets $V \subseteq U$ and $e_{U|V} = 0$ for all open subsets $V \subseteq X \setminus U$. Show that $U \mapsto e_U$ yields a bijection

$$\mathrm{OC}(X) \leftrightarrow \mathrm{Idem}(\Gamma(X, \mathcal{O}_X))$$

from the set of open and closed subsets of X to the set of idempotents in $\Gamma(X, \mathcal{O}_X)$.
ii) Show that $e_U e_{U'} = e_{U \cap U'}$ for $U, U' \in \mathrm{OC}(X)$.
iii) Prove that X is connected if and only if $\Gamma(X, \mathcal{O}_X)$ contains no idempotent $e \neq 0, 1$ if and only if $\Gamma(X, \mathcal{O}_X)$ is not isomorphic to $R_1 \times R_2$ for two non-zero rings R_1, R_2.

Exercise 3 (4 Points):
Let $m, n \geq 0$, and let $f \colon \mathbb{A}^n_{\mathbb{Z}} \to \mathbb{A}^m_{\mathbb{Z}}$ be a morphism of schemes. Denote by $x_1, \ldots, x_n$ the coordinates on $\mathbb{A}^n_{\mathbb{Z}}$, and by $y_1, \ldots, y_m$ the coordinates on $\mathbb{A}^m_{\mathbb{Z}}$.
i) Show that there exist unique polynomials $f_1, \ldots, f_m \in \mathbb{Z}[X_1, \ldots, X_n]$ such that for any scheme T the morphism f is via the Yoneda lemma given by the map

$$\Gamma(T, \mathcal{O}_T)^n = \mathbb{A}^n_{\mathbb{Z}}(T) \to \mathbb{A}^m_{\mathbb{Z}}(T) = \Gamma(T, \mathcal{O}_T)^m,$$

sending $(x_1, \ldots, x_n) \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$.
ii) Let K be a field, and let $x \in \mathbb{A}^n_{\mathbb{Z}}(K)$. Show that the map on tangent spaces

$$T_x f \colon K^n = T_x A^n_{\mathbb{Z}} \to T_{f(x)} A^m_{\mathbb{Z}} = K^m$$

is given by the Jacobi matrix

$$\Big(\frac{\partial f_i}{\partial X_j}(x)\Big)_{i=1,\ldots,m,j=1,\ldots,n},$$

where $\frac{\partial f_i}{\partial X_j} \in \mathbb{Z}[X_1, \ldots, X_m]$ denotes the partial derivative.

Exercise 4 (4 Points):

Let $a, b \in \mathbb{Z}$, and let $X = \operatorname{Spec}\big(\mathbb{Z}[x, y]/(y^2 - x^3 - ax - b)\big)$. Let K be an algebraically closed field of characteristic $p \neq 2, 3$. Show that the following are equivalent:

i) $p \nmid 4a^3 + 27b^2$;

ii) $\dim_K(T_x X) = 1$ for all $x \in X(K)$.

Here if $p = 0$, then $p \nmid 4a^3 + 27b^2$ simply means $4a^3 + 27b^2 \neq 0$ (by definition).

To be handed in during the lecture on: Wednesday, June 05.