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SS 2019

Algebraic Geometry I

6. Exercise sheet

Exercise 1 (4 Points):
Let X be a topological space and let $(U_i)_i$ be an open covering of X. For all i, let $\mathcal{F}_i$ be a sheaf on U_i. Assume that for each pair i, j of indices we are given isomorphisms $\varphi_{ij} : \mathcal{F}_{j|U_i \cap U_j} \to \mathcal{F}_{i|U_i \cap U_j}$ satisfying for all i, j, k the "cocycle condition"

$$\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk} \quad \text{on } U_i \cap U_j \cap U_k.$$

Show that there exists a sheaf $\mathcal{F}$ on X together with isomorphisms $\psi_i : \mathcal{F} \xrightarrow{\sim} \mathcal{F}_{|U_i}$ for all i, such that $\psi_i \circ \varphi_{ij} = \psi_j$ on $U_i \cap U_j$ for all i, j. Show that $\mathcal{F}$ and the ψ_i are determined up to unique isomorphism by these conditions.

Remark: We say that the sheaf $\mathcal{F}$ is obtained by *gluing the* $\mathcal{F}_i$ via the *gluing data* φ_{ij}, and that the sheaves on X form a *stack*.

Exercise 2 (4 Points):
Let p be a prime number, let $\mathbb{F}_p$ be the field with p elements, and let $\iota_p : \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ be the canonical morphism. We say that a ring A *has characteristic* p if $p \cdot 1 = 0$ in A. Let X be a scheme. Prove that the following conditions are equivalent:

(i) The ring $\Gamma(X, \mathcal{O}_X)$ has characteristic p.

(ii) For every open subset $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ has characteristic p.

(iii) The unique scheme morphism $X \to \operatorname{Spec} \mathbb{Z}$ factors through ι_p.

If the conditions are satisfied, we say that X *has characteristic* p. Show that in this case the morphism $X \to \operatorname{Spec} \mathbb{F}_p$ is unique.

Are these conditions equivalent to (iv)?

(iv) For all $x \in X$ the residue field $k(x)$ has characteristic p.

Exercise 3 (4 Points):
Let X be a scheme of characteristic p. Show that there exists a unique morphism of schemes $(F, F^\flat) : X \to X$ such that on topological spaces, $F = \mathrm{id}_X$, and for open subsets $U \subseteq X$, $F^\flat$ is the Frobenius endomorphism of $\Gamma(U, \mathcal{O}_X)$, $s \mapsto s^p$. This morphism is called the *absolute Frobenius morphism of* X.

Give an example where $(F, F^\flat)$ is not an isomorphism.

Exercise 4 (4 Points):
Let k be an algebraically closed field, $Z := V(T_1, \ldots, T_n) \subset \mathbb{A}_k^n$. Determine for which $n \geq 1$ the open subscheme $X := \mathbb{A}_k^n \setminus Z$ is affine.

To be handed in during the lecture on: Wednesday, May 29.