Prof. Dr. T. Richarz **SS 2019**
Dr. U. Buchholtz

Algebraic Geometry I

4. Exercise sheet

Exercise 1 (4 Points):
Let A be a ring.

1. Show that the topological space $\operatorname{Spec} A$ is T_0-separable.
2. Show that an open subset $U \subseteq \operatorname{Spec} A$ is quasi-compact if and only if it is the complement of a closed subset of the form $V(\mathfrak{a})$ with $\mathfrak{a}$ a finitely generated ideal in A.

Exercise 2 (4 Points):
Let A be a ring, and denote $A_{\text{red}} := A/\operatorname{rad}(0)$. Show that the following are equivalent:

(i) The topological space $\operatorname{Spec} A$ is irreducible.

(ii) The ring A_{red} is an integral domain.

Exercise 3 (4 Points):

1. Let $f\colon X \to Y$ be a morphism of topological spaces. Assume that X, Y are irreducible admitting generic points $\xi \in X$ and $\eta \in Y$, respectively, and that Y is T_0. Show that $f(X)$ is dense in Y if and only if $f(\xi) = \eta$.
2. Let (X_i, f_{ji}), be an inverse system of spectral spaces indexed by a cofiltered poset I (i.e. I^{op} is filtered) and for every $i \geq j$ in I the given morphism $f_{ji} : X_i \to X_j$ is quasi-compact, i.e., for every quasi-compact open $U \subseteq X_j$ its preimage $f_{ji}^{-1}(U)$ is again quasi-compact. Show that the inverse limit

$$X := \varprojlim_I X_i = \left\{ (x_i)_{i\in I} \in \prod\nolimits_{i\in I} X_i \,\middle|\, f_{ji}(x_i) = x_j \text{ for all } i, j \in I, i \geq j \right\}$$

with its inverse limit topology is again a spectral space and that each projection $X \to X_i$ is quasi-compact.

Exercise 4 (4 Points):

1. Show that every finite irreducible topological space admits a generic point. Deduce that finite T_0-spaces are spectral.
2. Express $\operatorname{Spec} \mathbb{Z}$ as an inverse limit of finite T_0-spaces.

To be handed in on: Wednesday, May 15.