Prof. Dr. T. Richarz
Dr. U. Buchholtz **SS 2019**

Algebraic Geometry I

2. Exercise sheet

Let k be an algebraically closed field and denote by $\mathbb{A}^n(k) = k^n$ the affine n-space over k together with its Zariski topology.

Exercise 1 (4 Points):

Let R be a ring (e.g., a k-algebra). For $f \in R$ and any R-module M, let $M[f^{-1}]$ be the localization at the multiplicative subset $\{1, f, f^2, \ldots\} \subset R$. Show the following statements used in the proof of Hilbert's Nullstellensatz:

1) The localization $R[f^{-1}]$ is zero if and only if $f \in R$ is nilpotent, i.e., there exists an integer $N \geq 1$ with $f^N = 0$.

2) For any ideal $\mathfrak{a} \subset R$ the localization $\mathfrak{a}[f^{-1}] \subset R[f^{-1}]$ is an ideal. Prove that $R[f^{-1}]/\mathfrak{a}[f^{-1}] = (R/\mathfrak{a})[\bar{f}^{-1}]$ where $\bar{f} \in R/\mathfrak{a}$ is the image of f under the quotient map $R \to R/\mathfrak{a}$.

Exercise 2 (4 Points):

Let $X \subseteq \mathbb{A}^n(k)$ be an affine algebraic set with coordinate ring $A := \mathcal{O}(X) = k[X_1, \ldots, X_n]/I(X)$. Let $f \in A$ be any element, and consider the set

$$D(f) := X \backslash V(f) = \{x = (x_1, \ldots, x_n) \in X \mid f(x_1, \ldots, x_n) \neq 0\}.$$

Show that

$$D(f) \subseteq \mathbb{A}^{n+1}(k),\ x \mapsto (x_1, \ldots, x_n, f(x_1, \ldots, x_n)^{-1})$$

is again an affine algebraic set with coordinate ring given by the localisation $A[f^{-1}]$.

Exercise 3 (4 Points):

Let X, Y be coordinates on $\mathbb{A}^2(k)$ and F a polynomial in $k[X] \subset k[X, Y]$. We denote by $\mathbb{C}$ the field of complex numbers. Consider the following maps:

1) $f_1 \colon \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C}),\ x \mapsto \exp(x)$

2) $f_2 \colon \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C}),\ x \mapsto \begin{cases} x+1 & \text{if } x \in \mathbb{Q}[i] \\ x & \text{else} \end{cases}$

3) $f_3 \colon \mathbb{A}^1(k) \to V(X^3 - Y^2),\ x \mapsto (x^2, x^3)$

4) $f_4 \colon V(F(X) - Y) \to \mathbb{A}^1(k),\ (x, y) \mapsto x$

Which maps are continuous for the Zariski topology? Which maps are morphisms of affine algebraic sets, and which are isomorphisms?

Exercise 4 (4 Points):

Let $\mathrm{char}(k) \neq 2$, and let $Z_1 = V(U(T-1) - 1)$ and $Z_2 = V(Y^2 - X^2(X+1))$ be closed subsets of $\mathbb{A}^2(k)$, with coordinates (T, U) and (X, Y), respectively. Show that $(t, u) \mapsto (t^2 - 1, t(t^2 - 1))$ defines a bijective morphism $Z_1 \to Z_2$ which is not an isomorphism.

To be handed in during the lecture on Monday, May 6.