(Pro-) Étale Cohomology
9. Exercise Sheet

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Winter Semester 18/19
12th December 2018

Homework

Exercise H27 (Profinite Groups) (12 points)

Let G be a topological group.

(a) Show that the following assertions are equivalent:
  (i) The underlying topological space of G is profinite.
  (ii) G is a limit of a diagram of finite discrete topological groups.

(b) Show that if one of the assertions above holds, we have

$$G = \lim_{H} G/H,$$

where the limit ranges over all normal open subgroups of G.

Exercise H28 (Galois group of finite fields) (3+3+3+3 points)

Let k be a finite field of characteristic $p > 0$ such that k has q elements. We fix an algebraic closure $\overline{k}$ of k and set $G := \mathrm{Gal}(\overline{k}/k)$. Let $\sigma \in G$ denote the element which is given by $x \mapsto x^q$.

(a) Show that $G \cong \hat{\mathbb{Z}} := \lim_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$.

(b) Show that G is topologically generated by σ, i.e. the subgroup of G generated by σ is dense in G.

Let H be an arbitrary topological group. A *(topological) H-module* is an abelian group M (considered as a topological group via the discrete topology) endowed with a continuous left action of H (i.e. the left action $H \times M \to M$ is continuous).

(c) Show that a topological G-module is the same as an abelian group M together with an automorphism $\alpha\colon M \to M$ which satisfies the following property: For all $m \in M$ there is an $N_m \in \mathbb{N}$ such that (N_m is the smallest natural number with) $\alpha^{N_m}(m) = m$.

(d) We consider the G-module $\overline{k}^{\times}$ (where $\varphi \in G$ acts on $m \in \overline{k}^{\times}$ by $x \mapsto \varphi(x)$). Show that in view of (c) this G-module corresponds to the pair $(\overline{k}^{\times}, \sigma)$ and that we cannot find $C \in \mathbb{N}$ such that $N_m \leq C$ for all $m \in \overline{k}^{\times}$.

Exercise H29 (Čech-cohomology and group cohomology) (3+2+3+4 points)

The goal of this exercise is to apply the comparison isomorphism of Čech-cohomology and sheaf-cohomology (Corollary 4.23) to the the results of Exercise H22 to find an explicit formula to calculate group cohomology.

In this exercise, we use the same notations as in Exercise H13 and Exercise H22. Let G be a group, A be a G-module and $h_A = \mathrm{Hom}_G(-,A)$ be the associated sheaf of abelian groups on the site (G-sets). We consider the covering $\mathscr{U} = (G \to e)$ of e in the site (G-sets).

(a) Show that
$\check{C}^q(\mathscr{U}, h_A) = \mathrm{Hom}_G(G^{q+1}, A) = \{\varphi\colon G^{q+1} \to A \mid \varphi(hg_0, \ldots, hg_q) = h\varphi(g_0, \ldots, g_q) \text{ for all } h, g_0, \ldots, g_q \in G\} =: C^q(G,A)$
and $d^{q-1}\colon C^{q-1}(G,A) \to C^q(G,A)$ is given by

$$(\varphi\colon G^q \to A) \mapsto (d^{q-1}(\varphi)\colon G^{q+1} \to A, (g_0, \ldots, g_q) \mapsto \sum_{i=0}^{q} (-1)^i \varphi(g_0, \ldots, \hat{g}_i, \ldots, g_q))$$

for all $q \geq 0$.

(b) We define $\mathscr{C}^q(G,A)$ to be the abelian group of all maps (of sets) $G^q \to A$. Show that for all $q \geq 0$ we have isomorphisms of abelian groups $C^q(G,A) \to \mathscr{C}^q(G,A)$ sending $(\varphi\colon G^{q+1} \to A) \in C^q(G,A)$ to

$$\begin{aligned} \psi\colon G^q &\to A, \\ (g_1, \dots, g_q) &\mapsto \varphi(1, g_1, g_1 \cdot g_2, \dots, g_1 \cdots g_q). \end{aligned}$$

(c) We define morphisms of abelian groups $\partial^{q-1}\colon \mathscr{C}^{q-1}(G,A) \to \mathscr{C}^q(G,A)$ sending $(\varphi\colon G^{q-1} \to A)$ to

$$\partial^{q-1}(\varphi)\colon G^q \to A,$$
$$(g_1, \dots, g_q) \mapsto g_1\varphi(g_2, \dots, g_q) + \sum_{i=1}^{q-1}(-1)^i \varphi(g_1, \dots, g_{i-1}, g_i \cdot g_{i+1}, g_{i+2} \dots, g_q) + (-1)^q \varphi(g_1, \dots, g_{q-1}).$$

Show that

$$\begin{array}{ccc} C^q(G,A) & \xrightarrow{d^q} & C^{q+1}(G,A) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \mathscr{C}^q(G,A) & \xrightarrow[\partial^q]{} & \mathscr{C}^{q+1}(G,A) \end{array}$$

commutes for all $q \geq 0$ and that the isomorphisms induce isomorphims of the cohomology groups of the complexes $C^\bullet(G,A) = (0 \to C^0(G,A) \xrightarrow{d^0} C^1(G,A) \xrightarrow{d^1} \dots)$ and $\mathscr{C}^\bullet(G,A) = (0 \to \mathscr{C}^0(G,A) \xrightarrow{\partial^0} \mathscr{C}^1(G,A) \xrightarrow{\partial^1} \dots)$

(d) Show that $\check{H}^q(\mathscr{U}, h_A) \cong H^q(e, h_A)$ for all $q \geq 0$ and conclude that $H^q(G,A) \cong H^q(\mathscr{C}^\bullet(G,A))$ for all $q \geq 0$.

Remark: If one does not want to work with derived functors, $H^q(G,A) := H^q(\mathscr{C}^\bullet(G,A))$ gives an ad hoc definition of group cohomology. This can often be found in the literature.