(Pro-) Étale Cohomology
8. Exercise Sheet

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Homework

Exercise H24 (Torsors and Cohomology) (3+3+6 points)
Let $\mathscr{C}$ be a finitely complete site with final object S and $\mathscr{G}$ be a sheaf of abelian groups on $\mathscr{C}$. In this exercise we construct a functorial bijection

$$H^1_{\mathscr{C}}(S,\mathscr{G}) \cong H^1_{\mathscr{C}}(\mathscr{G}) := \{\text{isomorphism classes of } \mathscr{G}\text{-torsors on } \mathscr{C}\}.$$

We describe the constructions of the respective maps:

(a) Let $\xi \in H^1_{\mathscr{C}}(S,\mathscr{G})$. We choose an injective map $\mathscr{G} \to \mathscr{I}$ of $\mathscr{G}$ into an injective sheaf of abelian groups $\mathscr{I}$ on $\mathscr{C}$ and set $\mathscr{F} := \mathscr{I}/\mathscr{G}$ to obtain a short exact sequence

$$0 \to \mathscr{G} \to \mathscr{I} \xrightarrow{\gamma} \mathscr{F} \to 0.$$

Show that there is a $q \in H^0_{\mathscr{C}}(S,\mathscr{F}) = \mathscr{F}(S)$ that maps to ξ under $\delta\colon H^0_{\mathscr{C}}(S,\mathscr{F}) \to H^1_{\mathscr{C}}(S,\mathscr{G})$ (where δ is the usual connection map from the associated long exact cohomology sequence). We define a subsheaf (of sets) $\mathscr{T} := \mathscr{T}_\xi \subseteq \mathscr{I}$ via

$$\mathscr{T}(X) = \{s \in \mathscr{I}(X) \mid \gamma_X(s) = q|_X\}$$

for any object X of $\mathscr{C}$. Show that this construction gives a well defined map

$$H^1_{\mathscr{C}}(S,\mathscr{G}) \to H^1_{\mathscr{C}}(\mathscr{G}),\ \xi \mapsto \mathscr{T}_\xi. \tag{1}$$

(b) Let $\mathscr{T}$ be a $\mathscr{G}$-torsor. We consider the sheaf of abelian groups $\mathbb{Z}[\mathscr{T}]$ on $\mathscr{C}$ which is defined to be the sheafification of the presheaf

$$\mathscr{C} \to (\mathrm{Ab}),\ U \mapsto \Big\{\sum_{i=0}^{k} n_i[s_i] \mid k \in \mathbb{N}_0, n_i \in \mathbb{Z}, s_i \in \mathscr{T}(U)\Big\},$$

where the right hand side is the abelian group of finite formal sums with coefficients in $\mathscr{T}(U)$.
Show that there is a natural map of sheaves of abelian groups on $\mathscr{C}$

$$\sigma\colon \mathbb{Z}[\mathscr{T}] \to \underline{\mathbb{Z}}_{\mathscr{C}},\ \sum_i n_i[s_i] \mapsto \sum_i n_i,$$

with $\ker(\sigma)$ generated by elements of the form $[s]-[s']$. Moreover, show that there is a map $a\colon \ker(\sigma) \to \mathscr{G}$ which sends $[s]-[s']$ to the unique $g \in \mathscr{G}(U)$ with $g \cdot s = s'$ for $U \in \mathrm{Ob}(\mathscr{C})$ and $s, s' \in \mathscr{T}(U)$. Show that we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\sigma) & \longrightarrow & \mathbb{Z}[\mathscr{T}] & \xrightarrow{\sigma} & \underline{\mathbb{Z}}_{\mathscr{C}} & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow & & \downarrow \mathrm{id} & & \\
0 & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{E} & \longrightarrow & \underline{\mathbb{Z}}_{\mathscr{C}} & \longrightarrow & 0,
\end{array}$$

where the left square is cocartesian, all vertical arrows are surjective and the rows are exact. The left square being cocartesian means that $\mathcal{E}$ is the pushout of the diagram

$$\begin{array}{ccc} \ker(\sigma) & \longrightarrow & \mathbb{Z}[\mathcal{T}] \\ \downarrow & & \\ \mathcal{G} & & . \end{array}$$

The long exact cohomology sequence associated to the lower exact sequence in particular gives a map

$$\delta\colon H^0_{\mathcal{C}}(S, \underline{\mathbb{Z}}_{\mathcal{C}}) \to H^1_{\mathcal{C}}(S, \mathcal{G}).$$

We set $\xi_{\mathcal{T}} := \delta(1)$. Show that this construction provides a well defined map

$$H^1_{\mathcal{C}}(\mathcal{G}) \to H^1_{\mathcal{C}}(S, \mathcal{G}),\ \mathcal{T} \mapsto \xi_{\mathcal{T}} \tag{2}$$

(c) Show that the maps defined in (a) and (b) are inverse to each other.

Solution: Stacks Project 21.5 First Cohomology and Torsors Stacks 12.5.13 for pushouts in abelian categories

Exercise H25 (Anabelian Cohomology) (4+4+4 points)

Let $\mathcal{C}$ be a finitely complete site with final object S and $\varphi : \mathcal{G} \to \mathcal{G}'$ a homomorphism of sheaves of groups on $\mathcal{C}$.

(a) Let $\mathcal{T}$ be a $\mathcal{G}$-torsor on $\mathcal{C}$. We let $\mathcal{G}$ act on $\mathcal{G}' \times \mathcal{T}$ via

$$(g, (g', t)) \mapsto (g'\varphi_U(g^{-1}), gt),$$

where $U \in \mathrm{Ob}(\mathcal{C}), g \in \mathcal{G}(U), g' \in \mathcal{G}'(U)$ and $t \in \mathcal{T}(U)$. We obtain a (set-valued) presheaf $\mathcal{C} \to$ (Sets) sending U to the set of $\mathcal{G}(U)$ orbits of $(\mathcal{G}' \times \mathcal{T})(U) = \mathcal{G}'(U) \times \mathcal{T}(U)$. We denote its sheafification by $\mathcal{G}' \times^{\mathcal{G}} \mathcal{T}$ and let $\mathcal{G}'$ act on this sheaf by left multiplication. Show that we obtain a morphism of pointed sets

$$H^1(\varphi)\colon H^1_{\mathcal{C}}(\mathcal{G}) \to H^1_{\mathcal{C}}(\mathcal{G}'),\ \mathcal{T} \mapsto \mathcal{G}' \times^{\mathcal{G}} \mathcal{T}.$$

Consider a short exact sequence of sheaves of groups on $\mathcal{C}$

$$1 \to \mathcal{G}' \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{G}'' \to 1.$$

(b) For $g'' \in \mathcal{G}''(S)$ we define a sheaf $\mathcal{T}_{g''}$ of sets on $\mathcal{C}$ via

$$\mathcal{T}_{g''}(U) := \{t \in \mathcal{G}(U) \mid \pi_U(t) = g''|_U\}$$

for $U \in \mathrm{Ob}(\mathcal{C})$. We let $\mathcal{G}'(U)$ act on $\mathcal{T}_{g''}(U)$ by $(g', t) \mapsto (\iota_U(g')t)$. Show that this defines a *connection map* of pointed sets

$$\delta\colon \mathcal{G}''(S) \to H^1_{\mathcal{C}}(\mathcal{G}'),\ g'' \mapsto \mathcal{T}_{g''}.$$

(c) Assume we are given a short exact sequence as above. Show that we have an exact sequence of pointed sets:

$$1 \to \mathcal{G}'(S) \xrightarrow{\iota(S)} \mathcal{G}(S) \xrightarrow{\pi(S)} \mathcal{G}''(S) \xrightarrow{\delta} H^1_{\mathcal{C}}(\mathcal{G}') \xrightarrow{H^1(\iota)} H^1_{\mathcal{C}}(\mathcal{G}) \xrightarrow{H^1(\pi)} H^1_{\mathcal{C}}(\mathcal{G}'')$$

Solution: Wedhorn Manifolds, Sheaves and Cohomology p. 148, Prop. 7.22.

Exercise H26 (Quadratic Spaces) (3+9 points)

Let k be a field with $\mathrm{char}(k) \neq 2$ and $n \in \mathbb{N}$. Let V be an n-dimensional k-vector space and $\beta\colon V \times V \to k$ be a non degenerate symmetric bilinear form on V. We denote the group of orthogonal automorphisms of V by $G := \mathrm{O}(V, \beta)$ and the small étale site $(\mathrm{Spec}(k))_{\text{ét}}$ of $\mathrm{Spec}(k)$ by $\mathcal{C}$. For an étale k-algebra R, we set

$$\mathcal{G}(\mathrm{Spec}(R)) := G \otimes_k R := \mathrm{O}(V \otimes_k R, \beta \otimes \mathrm{id}_R).$$

(a) Show that $\mathcal{G}$ defines a sheaf of groups on $\mathcal{C}$.

(b) Show that we have a bijection

$$\left\{ \begin{array}{l} \text{isomorphism classes of pairs } (V', \beta')\text{ , } V' \text{ is an } n\text{-dimensional } k\text{-vector space} \\ \text{and } \beta \text{ is a symmetric non degenerate bilinear form on } V' \end{array} \right\}$$

$$\cong$$

$$H^1_{\mathcal{C}}(\mathcal{G}).$$

Solution: Lineare Algebra II Skript Seite 65 Abschnitt 21