(Pro-) Étale Cohomology
7. Exercise Sheet

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Homework

Exercise H22 (Group cohomology) (2+2+2+2+2+2 points)

Let G be a group. A *(left) G-module* is an abelian group M endowed with a left action $a\colon G \times M \to M$ by group homomorphisms. A morphism of G-modules $f : M \to N$ is a G-equivariant group homomorphism. We denote the abelian category of G-modules by Mod_G. We consider the functor $F\colon \mathrm{Mod}_G \to (\mathrm{Ab})$ from the category of G-modules to the category of abelian groups which is given by

$$M \mapsto M^G := \{m \in M \mid g \cdot m = m \ \forall \ g \in G\}$$

(a) Show that F is left exact and that F is in general not right exact.

(b) Show that the equivalence in Exercise H13, induces an equivalence of categories between the category of sheaves of abelian groups on (G-sets) and Mod_G. In particular, Mod_G satisfies the properties $a), b)$ and $c)$ from Theorem 4.4. and there exists a right derived functor

$$RF\colon D(\mathrm{Mod}_G) \to D((\mathrm{Ab})).$$

The abelian group $H^i(G,A) := R^iF(A) := H^i(RF(A))$ is called *i-th cohomology group of G with coefficients in A*, where A is a G-module .

For a left G-module A, we denote the associated abelian sheaf on the category of G-sets $\mathrm{Hom}_G(\,\cdot\,,A)$ by h_A. Let e be a one-element set with the unique structure of a left G-set and denote the section functor from the category of abelian sheaves on left G-sets to the category of abelian groups $h_A \mapsto h_A(e)$ by Γ_e.

(c) Show that

$$H^i(G,A) = H^i(e,h_A) := R^i\Gamma_e(h_A) := H^i(R\Gamma_e(h_A)).$$

In particular, we see that group cohomology is a special case of sheaf cohomology.

Now let $H \subseteq G$ be a subgroup and denote the set of left cosets with the induced left G-set structure by G/H.

(d) Show that we have

$$H^i(G/H, h_A) = H^i(H,A)$$

for all $i \in \mathbb{Z}$ and all G-modules A (where on the right hand side we consider A as an H-module by restriction).

Let $\pi\colon G' \to G$ be a morphism of groups. This induces a continuous functor $\tilde{\pi}\colon (G\text{-Sets}) \to (G'\text{-Sets})$ and hence a morphism $\pi\colon (G'\text{-Sets}) \to (G\text{-Sets})$ of sites.

(e) Show that under the identification of abelian sheaves on (G-sets) with left G-modules, π_* is identified with the functor $A' \mapsto \mathrm{Hom}_{G'}(G,A')$ from the category of G'-modules to the category of G-modules. We denote this functor also by π_*. Show that for all G'-modules A', we get a spectral sequence

$$E_2^{p,q} = H^p(G, R^q\pi_*(A')) \Rightarrow E^{p+q} = H^{p+q}(G',A').$$

Let H be a normal subgroup of G and $\pi\colon G \to G/H$ the natural projection.

(f) Show that the associated spectral sequence from (e) is given by

$$E_2^{p,q} = H^p(G/H, H^q(H,A)) \Rightarrow E^{p+q} = H^{p+q}(G,A).$$

This spectral sequence is called *Hochschild-Serre spectral sequence*.

Exercise H23 (Mayer-Vietoris sequence) (8+4 points)

(a) Let $\mathscr{C}$ be a finitely complete site and $\mathscr{O}$ be a sheaf of rings on $\mathscr{C}$. Consider an object X of $\mathscr{C}$ and a covering $(f_i\colon U_i \to X)_{i=1,2}$ in $\mathscr{C}$ which consists of two morphisms. Assume that at least one of the morphisms f_1, f_2 is a monomorphism. Note $W := U_1 \times_X U_2$ and let $\mathscr{F}$ be a sheaf of $\mathscr{O}$-modules on $\mathscr{C}$. Show that there is a long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(X,\mathscr{F}) \to H^0(U_1,\mathscr{F}) \oplus H^0(U_2,\mathscr{F}) \to H^0(W,\mathscr{F}) \\ &\to H^1(X,\mathscr{F}) \to H^1(U_1,\mathscr{F}) \oplus H^1(U_2,\mathscr{F}) \to H^1(W,\mathscr{F}) \to H^2(X,\mathscr{F}) \to \cdots \end{aligned}$$

(b) Let S be an affine scheme. Calculate

$$H^i_{\text{ét}}(\mathbb{P}^1_S, \mathbb{G}_{a,S})$$

for all $i \geq 0$.

Remark: We will see in the upcoming lecture that $H^i_{\text{ét}}(Y,\mathscr{F}) = 0$ for all affine schemes Y, all quasi-coherent $\mathscr{O}_Y$-modules $\mathscr{F}$ and all $i > 0$. This fact might be used without a proof.