(Pro-) Étale Cohomology
6. Exercise Sheet

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Homework

Exercise H19 (Edge morphisms) (2+10 points)
Let $\mathscr{A}$ be an abelian category and $E_2^{p,q} \Rightarrow E^{p+q}$ be a spectral sequence in $\mathscr{A}$ such that $E_2^{p,q} = 0$ for all (p,q) with $p < 0$ or $q < 0$. We fix $m > 0$ and assume that $E_2^{p,q} = 0$ for all $p \in \mathbb{Z}$ and for all $0 < q < m$.

(a) Show that for all $i < m$ the edge morphisms $E_2^{i,0} \to E^i$ are isomorphisms.

(b) Show that the sequence

$$0 \to E_2^{m,0} \xrightarrow{\text{edge}} E^m \xrightarrow{\text{edge}} E_2^{0,m} \xrightarrow{\delta} E_2^{m+1,0} \xrightarrow{\text{edge}} E^{m+1}$$

is exact, where δ is defined by

$$E_2^{0,m} \xrightarrow{\sim} E_{m+1}^{0,m} \xrightarrow{d_{m+1}^{0,m}} E_{m+1}^{m+1,0} \xrightarrow{\sim} E_2^{m+1,0}.$$

Exercise H20 (Morphisms of spectral sequences) (4+8 points)
Let $\mathscr{A}$ be an abelian category.

(a) Define the notion of a morphism of spectral sequences in $\mathscr{A}$.

(b) Let $f : (E_2^{p,q} \Rightarrow E^{p+q}) \to ((E')_2^{p,q} \Rightarrow (E')^{p+q})$ be a morphism of spectral sequences in $\mathscr{A}$. Assume that the maps

$$E_2^{p,q} \to (E')_2^{p,q}$$

are isomorphisms for all $p, q \in \mathbb{Z}$. Show that f is an isomorphism of spectral sequences in $\mathscr{A}$.

Exercise H21 (K-injectivity) (12+6* points)
Let $\mathscr{A}$ be an abelian category, and set $\mathscr{K} := K(\mathscr{A})$. Let I be an object of $\mathscr{K}$. Show that the following assertions are equivalent:

(a) I is right F-acyclic for every triangulated functor $F : \mathscr{K} \to \mathscr{E}$.

(b) For every morphism $f : X \to I$ and quasi-isomorphism $s : X \to Y$ (both in $\mathscr{K}$), there exists $g : Y \to I$ in $\mathscr{K}$ with $g \circ s = f$.

(c) For every morphism $f : X \to I$ and quasi-isomorphism $s : X \to Y$ (both in $\mathscr{K}$), there exists a unique $g : Y \to I$ in $\mathscr{K}$ with $g \circ s = f$.

(d) Every quasi-isomorphism $s : I \to Y$ in $\mathscr{K}$ has a left inverse.

*(e) One has $\mathrm{Hom}_{\mathscr{K}}(X, I) = 0$ for every exact complex X in $\mathscr{K}$.

If I satisfies these equivalent conditions, then I is called *K-injective*.