(Pro-) Étale Cohomology
5. Exercise Sheet

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Homework

Exercise H16 (Sheaf of relative differentials) (8+4 points)

Let S be a scheme and $f : X \to S$ be an S-scheme. We consider the presheaf F on $X_{\text{ét}}$ which sends an étale X-scheme $U \to X$ to $\Omega_{U/S}(U)$.

(a) Let $j : U \to X$ be an étale X-scheme. Show that we have $\Omega_{U/S} = j^*(\Omega_{X/S})$.

(b) Show that F defines a sheaf on $X_{\text{ét}}$.

It might be helpful to consult Proposition (1.9) for part (a).

Solution:

(a) Stacks project 28.32.16

(b) Script 3.29

Exercise H17 (Artin-Schreier-Sequence) (2+6+4 points)

Let p be a prime number and S be a scheme over $\mathrm{Spec}(\mathbb{F}_p)$. We consider the additive group scheme $\mathbb{G}_{a,S}$ over S as a sheaf of abelian groups on $(Sch/S)_{\text{ét}}$. Consider the endomorphism $\alpha : \mathbb{G}_{m,S} \to \mathbb{G}_{m,S}$ which is given by $x \mapsto x^p - x$.

(a) Show that $\ker(\alpha) \cong \underline{\mathbb{Z}/p\mathbb{Z}}_S \in \mathrm{Sh}((Sch/S)_{\text{ét}})$.

(b) Show that

$$0 \to \ker(\alpha) \to \mathbb{G}_{a,S} \xrightarrow{\alpha} \mathbb{G}_{a,S} \to 0$$

is an exact sequence in $\mathrm{Sh}((Sch/S)_{\text{ét}})$.

(c) Show that in general the respective assertion of (b) is wrong, if we consider

$$\ker(\pi) \to \mathbb{G}_{a,S} \xrightarrow{\alpha} \mathbb{G}_{a,S}$$

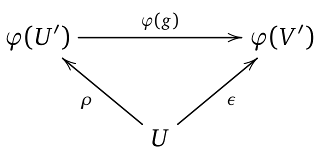
as a sequence of Zariski sheaves.

Exercise H18 (A left adjoint for φ^p) (2+2+2+3+3 points)

Let $\varphi : \mathscr{C} \to \mathscr{D}$ be a functor of small categories. We consider the induced functor $\varphi^p : \mathrm{PSh}(\mathscr{D}) \to \mathrm{PSh}(\mathscr{C})$ sending $\mathscr{F}$ to $\mathscr{F} \circ \varphi$. For an object U of $\mathscr{D}$ we define the category I_U to be the category whose objects consist of pairs (U', ρ), where U' is an object of $\mathscr{C}$ and $\rho : U \to \varphi(U')$ is a morphism in $\mathscr{D}$, and for which a morphism

$$(U', \rho) \to (V', \epsilon)$$

is a morphism $g : U' \to V'$ in $\mathscr{C}$ such that the diagram



commutes. For a presheaf $\mathscr{F}$ of $\mathscr{C}$ and an object U of $\mathscr{D}$, we define

$$(\varphi_p \mathscr{F})(U) := \mathrm{colim} \mathscr{F}(U')$$

where the colimit ranges over all (U', ρ) in I_U^{opp}.

(a) Show that this defines a functor $\varphi_p : \mathrm{PSh}(\mathscr{C}) \to \mathrm{PSh}(\mathscr{D})$.

(b) Show that the natural transformation $\varphi_p \varphi^p \to \mathrm{id}_{\mathrm{PSh}(\mathscr{D})}$ induces a map

$$\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \varphi^p \mathscr{G}) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{D})}(\varphi_p \mathscr{F}, \mathscr{G}),$$

for all $\mathscr{G} \in \mathrm{PSh}(\mathscr{D})$ and $\mathscr{F} \in \mathrm{PSh}(\mathscr{C})$, which is functorial in $\mathscr{F}$ and $\mathscr{G}$.

(c) For any $U' \in \mathscr{C}$ there is a canonical object $(U', \mathrm{id}_{\varphi(U')})$ of $I_{\varphi(U')}$. Use such objects to define a morphism of presheaves $\mathscr{F} \to \varphi^p \varphi_p \mathscr{F}$ for $\mathscr{F} \in \mathrm{PSh}(\mathscr{C})$. Use this morphism to define a functorial map

$$\mathrm{Hom}_{\mathrm{PSh}(\mathscr{D})}(\varphi_p \mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \varphi^p \mathscr{G}),$$

where $\mathscr{G} \in \mathrm{PSh}(\mathscr{C})$.

(d) Show that φ^p and φ_p are adjoint to each other.

(e) Now let $\mathscr{C}$ and $\mathscr{D}$ be sites and $\varphi : \mathscr{C} \to \mathscr{D}$ a continuous functor. We set $\varphi^s := \varphi^p|_{\mathrm{Sh}(\mathscr{D})} : \mathrm{Sh}(\mathscr{D}) \to \mathrm{Sh}(\mathscr{C})$. Show that φ^s is well defined and that it has a left adjoint, which we will denote by φ_s.