(Pro-) Étale Cohomology
4. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
Prof. Dr. Torsten Wedhorn
Timo Henkel

Winter Semester 18/19
9th November 2018

Homework

Exercise H13 (G-sets) (2+2+2+3+3 points)
Let G be a group. We consider the category (G-Sets) whose objects are sets endowed with a left action of G and whose morphisms are G-equivariant maps. We denote by ${}_GG$ the G-set whose underlying set is the underlying set of G and whose left G-action is the group multiplication. For a G-set X, a family of morphisms $(f_i : X_i \to X)_{i\in I}$ is called a covering of X, if $\bigcup_{i\in I} f_i(X_i) = X$.

(a) Show that this defines a Grothendieck topology on (G-Sets).

The map $\mathrm{Hom}_{G\text{-Sets}}({}_GG, {}_GG) \to G^{\mathrm{opp}}$ given by $\varphi \mapsto \varphi(1)$ defines an isomorphism of groups, whose inverse is given by $g \mapsto R_g$, where $R_g : {}_GG \to {}_GG$ maps x to xg . For $\mathscr{F}$ being a presheaf on (G-Sets), $g \in G$ and $s \in \mathscr{F}({}_GG)$ we set $g \cdot s := \mathscr{F}(R_g)(s)$.

(b) Show that $\mathscr{F} \mapsto \mathscr{F}({}_GG)$ defines a functor from the category of presheaves on (G-Sets) to the category (G-Sets), where $\mathscr{F}({}_GG)$ is endowed with the G-set structure defined above.

For a G-set X let $\mathscr{F}_X$ be the corresponding representable functor, i.e. the presheaf on (G-Sets) given by

$$\mathscr{F}_X(U) = \mathrm{Hom}_{G\text{-Sets}}(U, X).$$

(c) Show that $X \mapsto \mathscr{F}_X$ defines a functor from the category of G-sets to the category of sheaves on (G-Sets).

(d) Show that for a presheaf $\mathscr{F}$ on (G-Sets), the sheaf $\mathscr{F}_{\mathscr{F}({}_GG)}$ is the sheafification of $\mathscr{F}$.

(e) Show that the functors described above give an equivalence between the categories of sheaves on (G-Sets) and the category (G-Sets).

Exercise H14 (Covering maps and the fundamental groupoid) (2+2+3+3+2 points)
Let X be a path connected topological space such that there exists a universal covering map $\tilde{X} \to X$ (e.g. X is a connected topological manifold or more generally X is path connected, locally path connected and semilocally simply connected). We consider the category $\mathscr{C}$ of covering maps of X. This means the objects of $\mathscr{C}$ are given by $X' = (X', f)$, where X' is a (possibly empty) topological space and $f : X' \to X$ is a continuous map such that locally on X we have $X' \cong X \times F$, for a (possibly empty) discrete topological space F. The morphisms in $\mathscr{C}$ are the obvious ones. For a covering map $X' \to X$ in $\mathscr{C}$ we say that a family $(p_i : X_i \to X')_{i\in I}$ of morphisms in $\mathscr{C}$ is a covering of X', if $\bigcup_{i\in I} p_i(X_i) = X'$.

(a) Show that $\mathscr{C}$ has a final object and that fibre products exist in $\mathscr{C}$.

(b) Show that the notion of a covering defined above endows $\mathscr{C}$ with a Grothendieck topology. We denote the corresponding site by $(\mathscr{C}, \mathscr{T})$.

(c) Let x be an element of X. Recall (for example from [1] §3.4) that the following categories are equivalent:
  (i) The category of functors $\Pi(X) \to$ (Sets).
  (ii) The category ($\pi_1(X, x)$-Sets).
  (iii) The category $\mathscr{C}$.

(d) Show that the equivalence above induces an equivalence of sites between $(\mathscr{C}, \mathscr{T})$ and ($\pi_1(X, x)$-Sets) endowed with the Grothendieck topology defined in exercise H13.

(e) Deduce that a presheaf on $(\mathscr{C}, \mathscr{T})$ is a sheaf if and only if it is representable.

Remark: $\Pi(X)$ denotes the fundamental groupoid of X, i.e. the category whose objects are the elements of X and whose morphisms from x to y for $x, y \in X$ are given by the homotopy classes $[\gamma]$ of continuous maps $\gamma\colon [0,1] \to X$ with start point x and end point y. In particular, the automorphism group of an object x of $\Pi(X)$ is given by $\pi_1(X,x)^{\mathrm{opp}}$.

Exercise H15 (Coverings for different topologies) (2+2+2+2+4 points)

(a) For each of the following families of morphisms of schemes, decide if it is a covering for the Zariski (resp. fppf, resp. étale) topology and if it is an fpqc covering.

  (i) $X \to X$ being an isomorphism.

  (ii) $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ for $K \supseteq k$ being a field extension.

  (iii) $(\mathrm{Spec}(\kappa(x)) \to X)_{x \in X}$ for a scheme X.

  (iv) $(\mathrm{Spec}(\mathcal{O}_{X,x}) \to X)_{x \in X}$ for a scheme X.

(b) Let A be a noetherian ring and $f \in A$ a non zero-divisor. We denote the f-adic completion of A, which is defined as $\lim_{n \in \mathbb{N}}(A/(f^n))$, by $\hat{A}$. One can show that the canonical map $A \to \hat{A}$ is always flat in this case (which is not necessarily true if A is not noetherian).

  (i) Show that $\mathrm{Spec}(A_f) \sqcup \mathrm{Spec}(\hat{A}) \to \mathrm{Spec}(A)$ defines an fpqc covering of $\mathrm{Spec}(A)$.

  (ii) Assume that we are given a presheaf $\mathscr{F}$ on the category of schemes over $\mathrm{Spec}(A)$, such that $\mathscr{F}$ satisfies the sheaf condition for every fpqc covering. Show that we have a canonical bijection

$$\mathscr{F}(\mathrm{Spec}(A)) \cong \{(x,y) \in \mathscr{F}(\mathrm{Spec}(A_f)) \times \mathscr{F}(\mathrm{Spec}(\hat{A})) \mid \tilde{\pi}_1(x) = \tilde{\pi}_2(y)\}$$

  where $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are induced by the natural maps $\pi_1\colon A_f \to A_f \otimes_A \hat{A}$, resp. $\pi_2\colon \hat{A} \to A_f \otimes_A \hat{A}$ are the natural maps.

  (iii) Let K be a field, $A = K[T]$ and $f = T$. What are the rings $\hat{A}$ and $A_f \otimes_A \hat{A}$?

References

[1] Tammo tom Dieck. *Algebraic Topology*. European Mathematical Society, 2008