(Pro-) Étale Cohomology
3. Exercise Sheet

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Homework

Exercise H9 (Clopen subschemes) (12 points)
Let X be a scheme. We define

$$\mathrm{Clopen}(X) := \{Z \subseteq X \mid Z \text{ open and closed subscheme of } X\}.$$

Recall that $\mathrm{Clopen}(X)$ is in bijection to the set of idempotent elements of $\mathscr{O}_X(X)$.
Let $X \to S$ be a morphism of schemes. We consider the functor $F_{X/S}$ from the category of S-schemes to the category of sets, given by

$$F_{X/S}(T \to S) = \mathrm{Clopen}(X \times_S T).$$

Now assume that $X \to S$ is a finite locally free morphism of schemes. Show that $F_{X/S}$ is representable by an affine étale S-scheme which is of finite presentation over S.

Exercise H10 (Lifting criteria) (12 points)
Let $f : X \to S$ be a morphism of schemes which is locally of finite presentation. Consider the following diagram of S-schemes:

$$\begin{array}{ccc} T_0 & \longrightarrow & X \\ \downarrow & & \downarrow f \\ T & \longrightarrow & S \end{array} \tag{1}$$

Let $\mathscr{C}$ be a class of morphisms of S-schemes. We say that $\mathscr{C}$ satisfies the $\exists_{\leq 1}$-*lifting property* (resp. $\exists!$-*lifting property*) with respect to f, if for all morphisms $T_0 \to T$ in $\mathscr{C}$ and for all diagrams of the form (1) there exists at most (resp. exactly) one morphism of S-schemes $T \to X$ which makes the diagram commutative.
Let

$$\mathscr{C}_1 := \{f : T_0 \to T \text{ closed immersion of } S\text{-schemes} \mid f \text{ is given by a locally nilpotent ideal}\}$$
$$\mathscr{C}_2 := \{f : T_0 \to T \text{ closed immersion of } S\text{-schemes} \mid T \text{ is affine and } T_0 \text{ is given by a nilpotent ideal}\}$$
$$\mathscr{C}_3 := \{f : T_0 \to T \text{ closed immersion of } S\text{-schemes} \mid T \text{ is the spectrum of a local ring and } T_0 \text{ is given by an ideal } I \text{ with } I^2 = 0\}$$

Show that the following assertions are equivalent:

(i) $\mathscr{C}_1$ satisfies the $\exists_{\leq 1}$-lifting property (resp. $\exists!$-lifting property) with respect to f.
(ii) $\mathscr{C}_2$ satisfies the $\exists_{\leq 1}$-lifting property (resp. $\exists!$-lifting property) with respect to f.
(iii) $\mathscr{C}_3$ satisfies the $\exists_{\leq 1}$-lifting property (resp. $\exists!$-lifting property) with respect to f.

You can use the following fact without a proof: For an S scheme $Y \to S$ which is locally of finite presentation and a filtered system of affine S-schemes $(\mathrm{Spec}(A_i) \to S)_{i \in I}$ the canonical map

$$\mathrm{colim}_{i\in I}(\mathrm{Hom}_S(\mathrm{Spec}(A_i), Y)) \to \mathrm{Hom}_S(\mathrm{Spec}(\mathrm{colim}_{i\in I} A_i), Y)$$

is bijective.

Exercise H11 (Vanishing differentials for weakly étale maps) (4+8 points)

(a) Let R be a ring and $I \subseteq R$ be an ideal. Show that R/I is a flat R-module if and only if for all ideals $J \subseteq R$ we have $J \cap I = J \cdot I$.

(b) Let $f : X \to S$ be a weakly étale morphism of schemes. Show that $\Omega_f = 0$.

Exercise H12 (Properties of weakly étale) (6+6 points)

(a) Let $f : X \to S$ be a morphism of schemes. Show that the following assertions are equivalent:

(i) f is weakly étale.

(ii) The map $\mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x}$ is weakly étale for every $x \in X$.

(b) Let

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

be a commutative and Cartesian diagram of schemes such that g is faithfully flat. Show that f' is weakly étale if and only if f is weakly étale.