(Pro-) Étale Cohomology
2. Exercise Sheet

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Homework

Exercise H5 (Monic polynomials and étale morphisms) (6+6 points)
Let A be a ring.

(a) Let $f \in A[X]$ be a monic polynomial of degree $n \in \mathbb{N}_0$. Show that $A[X]/(f)$ is a free A-module of rank n.

(b) Show that for $n \in \mathbb{N}$ and $a \in A$ the morphism $\operatorname{Spec}(A[X]/(X^n - a)) \to \operatorname{Spec}(A)$ is étale if and only if n and a are units in A or $n = 1$.

Exercise H6 (Faithfully flatness) (6+4+2 points)

(a) Let R be a ring and $\varphi : R \to A$ be a flat R-algebra. Show that the following assertions are equivalent:
   (i) $\operatorname{Spec}(A) \to \operatorname{Spec}(R)$ is surjective.
   (ii) For all maximal ideals $\mathfrak{m}$ of R the ideal $\varphi(\mathfrak{m})A$ is not equal to A.
   (iii) If M is a non-zero R-module, then $M \otimes_R A \neq 0$.
   (iv) For every R-module M the map $M \to M \otimes_R A,\ x \mapsto x \otimes 1$ is injective.
   (v) We have $\varphi^{-1}(\varphi(\mathfrak{a})A) = \mathfrak{a}$ for all ideals $\mathfrak{a} \subseteq R$.

   If A satisfies these properties, A is called *faithfully flat over R*.

(b) Let $f : X \to S$ be a flat morphism of schemes with $x \in X$ and $s := f(x) \in S$. Show that the induced map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is faithfully flat, i.e. it makes $\mathcal{O}_{X,x}$ a faithfully flat $\mathcal{O}_{S,s}$-algebra.

(c) Let $f : X \to S$ be a flat morphism of schemes. Show that f is *generizing*, i.e. for all $x \in X$ and all generizations y' of $f(x)$ in S, there is a generization x' of x in X such that $f(x') = y'$. (For a topological space X and $x, y \in X$, we say that x is a generization of y if $y \in \overline{\{x\}}$.)

Exercise H7 (Ramification) (8+4 points)
Let A be a Dedekind domain, $K = \operatorname{Quot}(A)$ be its quotient field, $K \hookrightarrow L$ a finite field extension and $B \subseteq L$ the integral closure of A in L. One can show that B is again a Dedekind domain (using the theorem of Krull-Akizuki which may be used without a proof). We assume that B is a finite A-algebra (this condition is for example satisfied, if L is a separable field extension of K).
Remark: Dedekind domains are integral domains in which any non-zero ideal can uniquely (up to order) be written as a product of prime ideals.

(a) Show that $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is étale if and only if $\Omega_{B/A} = 0$ and that these two equivalent conditions imply that for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$ the ramification indices of $\mathfrak{p}$ in L (i.e. the exponents e_i of the unique factorization $\mathfrak{p}B = \mathfrak{q}_1^{e_1} \cdot \ldots \cdot \mathfrak{q}_l^{e_l}$) are equal to 1.

(b) Under which conditions does the inverse implication also hold?

Exercise H8 (Étale or not étale, that is the question) (6+1+1+1+1+1+1 points)

(a) Let R be a ring and $S = \operatorname{Spec}(R)$ the associated affine scheme. For any $n \in \mathbb{N}$ consider the functor from the category of affine S-schemes to the category of sets which sends an affine S-scheme $X = \operatorname{Spec}(A)$ to the set $\{x \in A \mid x^n = 1\}$. Show that this functor is representable by an affine S-scheme, which we denote by $\mu_{n,R}$. Under which conditions on R and n is the structure morphism $\mu_{n,R} \to \operatorname{Spec}(R)$ étale?

(b) Determine if the following morphisms of schemes are étale:

(i) $\mathbb{P}^1_R \to \mathrm{Spec}(R)$ where R is a non-zero ring.
(ii) Open immersions of schemes $U \hookrightarrow X$.
(iii) The closed immersions $\mathrm{Spec}(\mathbb{Z}/N\mathbb{Z}) \to \mathrm{Spec}(\mathbb{Z})$ for $N \in \mathbb{N}_0$.
(iv) $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$.
(v) $\mathrm{Spec}(\mathbb{R}) \to \mathrm{Spec}(\mathbb{Q})$.
(vi) $\mathrm{Spec}(\mathbb{Z}_{(p)}) \to \mathrm{Spec}(\mathbb{Z})$ for $p \in \mathbb{Z}$ a prime number.