(Pro-) Étale Cohomology
14. Exercise Sheet

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Homework

Exercise H24 (Étale cohomology of the projective space)

(a) Let X be a scheme and $Z \xrightarrow{i} X$ a closed immersion with open complement $U \xrightarrow{j} X$. Show that for any $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})$ there exists a short exact sequence

$$0 \to j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \to 0.$$

Let k be an algebraically closed field and $m \in \mathbb{N}$ a natural number that is invertible in k and $n \in \mathbb{N}$ an arbitrary natural number. We fix a primitive m-th root of unity $\zeta \in k$ to fix an isomorphism $\mu_m \cong \mathbb{Z}/m\mathbb{Z} =: \Lambda$.

(b) Show that for the compact cohomology we have $H^q_c(\mathbb{A}^n_k, \underline{\Lambda}_{\mathbb{A}^n_k}) \cong \Lambda$ for $q = 2n$ and $H^q_c(\mathbb{A}^n_k, \underline{\Lambda}_{\mathbb{A}^n_k}) = 0$ for all $q \neq 2n$.

(c) Let X be a proper scheme over k. Moreover, assume that there is a set theoretic stratification $X = \bigsqcup_{i \in I} X_i$ with $X_i \subseteq X$ locally closed subsets and $X_i \cong \mathbb{A}^{m_i}_k$ as schemes for suitable $m_i \in \mathbb{N}_0$. Use (a) and (b) to show that

$$H^q(X, \underline{\Lambda}_X) = \bigoplus_{i \in I,\ 2m_i = q} \Lambda$$

(d) Compute $H^q(\mathbb{P}^n_k, \underline{\Lambda}_{\mathbb{P}^n_k})$ for all $q \in \mathbb{Z}$ and $n \in \mathbb{Z}^{\geq 1}$.

Hints for:

(b) You can for example prove (b) by induction on n. For the case $n = 1$ use (a). To proceed, you can by induction calculate $H^q(\mathbb{A}^n_k, \underline{\Lambda}_{\mathbb{A}^n_k})$ by using the following result as a black box: Let S be a qcqs scheme and let $f : \mathbb{A}^1_S \to S$ be the affine line over S. Then $R^q f_! \underline{\Lambda}_{\mathbb{A}^1_S} \cong \underline{\Lambda}_S$ for $q = 2$ and $R^q f_! \underline{\Lambda}_{\mathbb{A}^1_S} = 0$ for $q \neq 2$.
To conclude, you can (again without a proof) use the following theorem which is called Poincaré duality: Let k be an algebraically closed field and X be a separated and smooth k-scheme of finite type of pure dimension d. Then

$$H^q_c(X, \underline{\Lambda}_X) \cong H^{2d-q}(X, \underline{\Lambda}_X).$$

This isomorphism depends on the choice of ζ.

(c) You can do induction on $n := \dim(X)$, where you use the long exact cohomology sequence associated to the sequence of (a) for $U = \bigsqcup_{i \in I_{top}} X_i$ (for $I_{top} := \{i \in I \mid X_i \cong \mathbb{A}^n_k\}$) and apply the results from (b).

Exercise H25

Let S and X be schemes and $f : X \to S$ a separated morphism of finite type. We consider the following Cartesian diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S. \end{array}$$

Let $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})$ be a torsion sheaf and $q \in \mathbb{N}_0$ a natural number. Show that the natural map

$$g^*(R^q f_! \mathscr{F}) \to R^q f'_!(g'^* \mathscr{F})$$

is an isomorphism.
Hint: Deduce this from the case where $f : X \to S$ is proper.