(Pro-) Étale Cohomology
13. Exercise Sheet

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Homework

Exercise H36 (Blow-up) (6+6 points)
Let $0 \leq i \leq n$ be integers and S be a scheme. We consider the following set-valued presheaf F on the category of S-schemes, defined by

$$(X \to S) \mapsto \{(L, L') \mid L \subseteq \mathcal{O}_X^{n+1}, L' \subseteq \mathcal{O}_X^{i+1} \text{ locally direct summands of rank 1 such that } \pi_{i,X} \text{ maps } L \text{ to } L'\}$$

where $\pi_{i,X} : \mathcal{O}_X^{n+1} \to \mathcal{O}_X^{i+1}$ is the projection to the first $i+1$ coordinates. Recall that $\mathbb{P}_S^i$ represents the presheaf on the category of S-schemes

$$(X \to S) \mapsto \{L \subseteq \mathcal{O}_X^{i+1} \mid L \text{ is a locally direct summand of rank 1}\}.$$

We consider the morphism of functors $F \to \mathbb{P}_S^i$, $(L, L') \mapsto L'$.

(a) Show that there exists an open covering $(U_i)_{i \in I}$ of $\mathbb{P}_S^i$ such that $U_i \times_{\mathbb{P}_S^i} F \cong \mathbb{P}_{U_i}^{n-i}$ as functors on the category of U_i-schemes. Moreover, conclude that F is representable by a smooth projective scheme over $\mathbb{P}_S^i$. We denote this scheme by $\mathrm{Bl}_{\Delta_i}(\mathbb{P}_S^n)$.

(b) Show that the fibres of the morphism $\mathrm{Bl}_{\Delta_i}(\mathbb{P}_S^n) \to \mathbb{P}_S^n$, $(L, L') \mapsto L$ have dimension less or equal to i.

Remark: $\mathrm{Bl}_{\Delta_i}(\mathbb{P}_S^n)$ is a blow-up of $\mathbb{P}_S^n$ (cf. [1] 13.19).

Solution: Wir benutzen, dass L, L' lokal direkte Summanden (Def. vgl. Prop. 8.10 in [GW]) sind, indem wir benutzen können, dass die jeweiligen Quotienten $\mathcal{O}_X^{n+1}/L$ wieder Vektorbündel sind.

Exercise H37 (Constructible Sheaves) (8+4 points)

(a) Let X be a topological space that can be written as a finite union $X = T_1 \cup \ldots \cup T_n$ of constructible subsets. Show that there exists a finite partition $X = \bigsqcup_{i \in I} X_i$ into locally closed and constructible subsets such that for all $i = 1, \ldots, n$ we have $T_i = \bigcup_{j \in J_i} X_j$ for some subset $J_i \subseteq I$.

(b) Let X be a quasi-compact and quasi-separated scheme and $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$ be a sheaf on the étale site of X. Show that the following assertions are equivalent:

  (i) $\mathscr{F}$ is constructible.

  (ii) There exists an open covering $X = \bigcup_{i=1}^n U_i$ such that $\mathscr{F}|_{U_i}$ is constructible.

  (iii) There exists a set theoretical decomposition $X = \bigsqcup_{i \in I} X_i$ into disjoint locally closed and constructible subsets such that I is finite and $\mathscr{F}|_{X_i}$ is locally constant constructible for all $i \in I$.

Solution:

(a) 09Y4

(b) 095E

X irred top. Raum mit generischem Punkt η. Dann ist eine lokal abgeschlossene Teilmenge von X, die η enthält bereits offen.

Exercise H38 (The category of constructible sheaves) (12 points)
Let X be a noetherian scheme, $\Lambda = \mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}_0$ and $\mathscr{F} \in (\Lambda\text{-Mod}(X_{\text{ét}}))$. Show that the following assertions are equivalent:

(i) $\mathscr{F}$ is constructible.

(ii) $\mathscr{F}$ is a noetherian object in the abelian category $(\Lambda\text{-Mod}(X_{\text{ét}}))$.

(iii) There exists a morphism $f : U \to V$ of quasi-compact objects in $X_{\text{ét}}$ such that

$$F \cong \operatorname{coker}((j_{V\to X})_! \underline{\Lambda}_V \xrightarrow{f_*} (j_{U\to X})_! \underline{\Lambda}_U).$$

Solution: 095N
09BH
In particular for any sub sheaf of a constructible sheaf is constructible.

Exercise H39 (Push forward of constructible sheaves) (2+2+3+3+2 points)
Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F} := \underline{\mathbb{Z}/n\mathbb{Z}}_X \in \text{Ab}(X_{\text{ét}})$. Decide in each of the following cases, whether $f_*\mathscr{F} \in \text{Ab}(Y_{\text{ét}})$ is constructible/locally constant /locally constant constructible:

(a) $\mathbb{A}^1_k \to \mathbb{A}^1_k, x \mapsto x^m$, for some field k and some $m \in \mathbb{Z}$ such that m is invertible in k.

(b) $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}, x \mapsto x^m$, for some field k and some $m \in \mathbb{Z}$ such that m is invertible in k.

(c) $\mathbb{A}^1_k \to \mathbb{A}^1_k, x \mapsto x^p - x$, for some field k of characteristic $p > 0$.

(d) $Y = \mathbb{A}^1_k$ for some field k and $X = \mathbb{A}^1_k$ with infinitely many origins.

(e) Y an arbitrary scheme and $X = \bigsqcup_{i\in I} Y$ for an infinite index set I.

Solution: Benutze Korollar 8.8 für einige Fälle. Dann ist die Garbe jeweils konstant und damit insbesondere alles andere. Bemerke: z.b. bei (a) ist die Abbildung gegeben durch $k[T] \to k[T], T \mapsto T^m$

References

[1] U. Görtz, T. Wedhorn. *Algebraic Geometry I*. Vieweg und Teubner, 2010