(Pro-) Étale Cohomology
12. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
Prof. Dr. Torsten Wedhorn
Timo Henkel

Winter Semester 18/19
23rd January 2019

Homework

Exercise H36 (Vanishing on stalks of closed points) (6+6 points)
Let k be a field and X a scheme which is of finite type over k. Moreover, let $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})$ be a sheaf of abelian groups on the étale site of X.

(a) Assume that k is algebraically closed. Show that $\mathscr{F} = 0$ if and only if $\mathscr{F}_x = 0$ for all $x \in X(k)$.

(b) Show that the assertion from (a) does not stay true, if k is only separably closed but not algebraically closed.

Exercise H37 (Higher direct images under finite morphisms) (6+6 points)
Let $f : X \to Y$ be a finite morphism of schemes and let $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})$ be a sheaf of abelian groups on the étale site of X.

(a) Show that $(R^q f_* \mathscr{F})_{\overline{y}} = 0$ for all $q > 0$ and all geometric points $\overline{y} \to Y$.

(b) What can be said about $(R^0 f_* \mathscr{F})_{\overline{y}}$ for a geometric point $\overline{y} \to Y$.

Remark: For (a) you can use the following lemma (without a proof):
Lemma 1. *(cf. Stacks Project Tag 03QJ) Let R be a henselian local ring and A be a finite R-algebra. Then A is a finite product of henselian local rings.*

Exercise H38 (Cohomology of the affine line) (5+4+3 points)
Let k be an algebraically closed field of characteristic $p > 0$.

(a) Assume that $n \in \mathbb{N}_0$ and p are coprime. Show that $H^q_{\text{ét}}(\mathbb{A}^1_k, \underline{\mathbb{Z}/n\mathbb{Z}}) = 0$ for all $q > 0$ and $H^0_{\text{ét}}(\mathbb{A}^1_k, \underline{\mathbb{Z}/n\mathbb{Z}}) = \mathbb{Z}/n\mathbb{Z}$.

(b) Show that $H^1_{\text{ét}}(\mathbb{A}^1_k, \underline{\mathbb{Z}/p\mathbb{Z}})$ is of infinite dimension and $H^q_{\text{ét}}(\mathbb{A}^1_k, \underline{\mathbb{Z}/p\mathbb{Z}}) = 0$ for all $q > 1$.

(c) Let X be a proper scheme over k. Show that $H^q_{\text{ét}}(X, \underline{\mathbb{Z}/p\mathbb{Z}})$ is of finite dimension for all $q \geq 0$.

Exercise H39 (Weil's Uniformization Theorem) (2+2+3+2+3 points)
Let C be a normal noetherian integral scheme of dimension 1 (e.g. C is an integral regular curve over some algebraically closed field k). Let $j : \eta \to C$ denote the generic point of C, $K := \mathscr{O}_{C,\eta}$ its function field and $|C| \subseteq C$ the subset of closed points. Moreover, for $c \in |C|$ let $\mathscr{O}_c$ denote the completion of $\mathscr{O}_{C,c}$ with respect to its maximal ideal and $K_c := \mathrm{Frac}(\mathscr{O}_c)$.

(a) Show that the only non trivial closed subsets of C are the finite subsets of $|C|$.

We define the *ring of finite adeles of C* by

$$\mathbb{A} := \mathbb{A}_C := \left\{ (x_c)_{c \in |C|} \in \prod_{c \in |C|} K_c \mid \text{almost all } x_c \in \mathscr{O}_c \right\}$$

as a subring of $\prod_{c \in |C|} K_c$.

(b) Show that $K \to \mathbb{A}, x \mapsto (x)_{c \in |C|}$ gives a well defined ring homomorphism.

We fix some open non empty subset $U \subseteq C$ and set $C \setminus U = \{c_1, ..., c_r\}$. Let $\mathscr{C}_U$ denote the category of triples

$$(E, (M_i)_{i=1}^r, (u_i)_{i=1}^r),$$

where E is a vector bundle over U, M_{c_i} is a free $\mathscr{O}_{c_i}$-module and $u_{c_i} : M_{c_i} \otimes_{\mathscr{O}_{c_i}} K_{c_i} \to E_\eta \otimes_K K_{c_i}$ is an isomorphism of K vector spaces for all $i = 1, ..., r$ (here E_η denotes the K vector space associated to the vector bundle $j_* E$ on η). A morphism of two objects $(E,(M_i)_{i=1}^r,(u_i)_{i=1}^r)$ and $(E',(M_i')_{i=1}^r,(u_i')_{i=1}^r)$ consist of a morphism of vector bundles $E\to E'$ and $\mathscr{O}_{c_i}$-linear maps from M_i to M_i' which are compatible with the u_i and u_i' for all $i=1,\dots,r$. Moreover, we denote the category of vector bundles on C by Fib. We obtain a natural functor

$$\mathrm{Fib}\to\mathscr{C}_U$$

which is induced by the cartesian diagram

$$\begin{array}{ccc}\operatorname{Spec}(\prod_{i=1}^r K_{c_i}) & \longrightarrow & \operatorname{Spec}(\prod_{i=1}^r \mathscr{O}_{c_i})\\ \downarrow & & \downarrow\\ U & \longrightarrow & C.\end{array}$$

Given a vector bundle $\mathscr{F}$ over C we obtain an object of $\mathscr{C}_U$ in the following way: Pull $\mathscr{F}$ back to U and $\operatorname{Spec}(\prod_{i=1}^r\mathscr{O}_{c_i})$ respectively and obtain the u_i by pulling both such bundles back to $\operatorname{Spec}(\prod_{i=1}^r K_{c_i})$. Since the canonical morphism $U\sqcup\operatorname{Spec}(\prod_{i=1}^r\mathscr{O}_{c_i})\to C$ is quasi-compact and faithfully flat, the functor described above is an equivalence of categories (this is a special case of faithfully flat descent, cf. [1] Theorem 14.66) that respects ranks.

(c) Now let $n\in\mathbb{N}$, $\emptyset\neq U=\operatorname{Spec}A\subseteq C$ open and affine and $C\setminus U=\{c_1,\dots,c_r\}$. Let $\mathscr{F}$ be a rank n vector bundle on C (with associated triple $(E,(M_i)_i,(u_i)_i)$) which can be trivialized over U. We fix such a trivialization

$$\psi: E\xrightarrow{\cong}(\mathscr{O}_C|_U)^n=\widetilde{A^n}.$$

Then for any $c\in\{c_1,\dots,c_r\}$ we have an isomorphism

$$\beta_c: M_c\otimes_{\mathscr{O}_c}K_c\xrightarrow{u_c}E_\eta\otimes_K K_c\xrightarrow{\psi\otimes\mathrm{id}}K^n\otimes_K K_c=(K_c)^n.$$

Show that we obtain the following bijections:

$$\begin{aligned}&\{\text{rank } n \text{ vector bundles } \mathscr{F} \text{ on } C \text{ with fixed trivialization } \psi \text{ on } U\}/\cong\\ &\longleftrightarrow\prod_{i=1}^r\{\text{free } \mathscr{O}_{c_i}\text{-submodules of } K_{c_i}^n \text{ of rank } n\}\\ &\longleftrightarrow\prod_{i=1}^r \mathrm{GL}_n(K_{c_i})/\mathrm{GL}_n(\mathscr{O}_{c_i}),\end{aligned}$$

where the first bijection is given by $\mathscr{F}\mapsto(\beta_{c_i}(M_i))_{i=1,\dots,r}$.

(d) Show that varying the trivializations leads to a bijection

$$\begin{aligned}&\{\text{rank } n \text{ vector bundles } \mathscr{F} \text{ on } C \text{ such that } \mathscr{F}|_U \text{ is trivial}\}/\cong\\ &\longleftrightarrow \mathrm{GL}_n(A)\setminus\mathrm{GL}_n(\prod_{i=1}^r K_{c_i})/\mathrm{GL}_n(\prod_{i=1}^r\mathscr{O}_{c_i})\end{aligned}$$

where the action of $\mathrm{GL}_n(A)$ on $\prod_{i=1}^r\mathrm{GL}_n(K_{c_i})$ is induced by the homomorphism $A\to\prod_{i=1}^r K_{c_i}, a\mapsto(a)_{i=1,\dots,r}$.

(e) Let $n\in\mathbb{N}$. Show that there is a bijection

$$\begin{gathered}\{\text{rank } n \text{ vector bundles on } C\}/\cong\\ \longleftrightarrow\\ \mathrm{GL}_n(K)\setminus\mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathscr{O}),\end{gathered}$$

where $\mathscr{O}:=\prod_{c\in|C|}\mathscr{O}_c$.

References

[1] U. Görtz, T. Wedhorn. *Algebraic Geometry I*. Vieweg und Teubner, 2010