(Pro-) Étale Cohomology
11. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics
Prof. Dr. Torsten Wedhorn
Timo Henkel

Winter Semester 18/19
16th January 2019

Homework

Exercise H33 (Stalks in the étale topology) (12 points)

Let S be a scheme and $\bar{s} \xrightarrow{i} S$ be a geometric point of S. Moreover, let $\mathscr{U}_{\text{ét}}(\bar{s})$ denote the cofiltered category of étale neighbourhoods of $\bar{s}$. For a presheaf $\mathscr{F} \in \text{PSh}(S_{\text{ét}})$ we set

$$\mathscr{F}_{\bar{s}} = \text{colim}_U\, \mathscr{F}(U),$$

where the colimit ranges over all $U \in \mathscr{U}_{\text{ét}}(\bar{s})^{\text{opp}}$. Recall that i induces a continuous functor $\mathscr{C} := S_{\text{ét}} \xrightarrow{\varphi} (\bar{s})_{\text{ét}} =: \mathscr{D}$ and functors (c.f. Exercise H18)

$$\varphi_p\colon \text{PSh}(\mathscr{C}) \to \text{PSh}(\mathscr{D}),$$
$$i^* = \varphi_s\colon \text{Sh}(\mathscr{C}) \to \text{Sh}(\mathscr{D}).$$

Let $\mathscr{F} \in \text{PSh}(S_{\text{ét}})$. Show that we have natural isomorphisms

$$\mathscr{F}_{\bar{s}} \cong (\varphi_p(\mathscr{F}))(\bar{s}) \cong (\varphi_p(\mathscr{F}))^\sharp(\bar{s}) \cong (i^*(\mathscr{F}^\sharp))(\bar{s}).$$

Exercise H34 (Henselization and strict Henselization) (1+3+4+3+1 points)

Let S be a scheme and $s \in S$. We consider the cofiltered category $\mathscr{U}_0(s)$ whose objects are pairs (U,u) where $g\colon U \to S$ is an étale S-scheme and $u \in U$ is a point with $g(u) = s$ and $\kappa(s) \cong \kappa(u)$. A morphism $(U,u) \to (U',u')$ in $\mathscr{U}_0(s)$ is a morphism $U \to U'$ mapping u to u'. We set

$$\mathscr{O}^h_{S,s} = \text{colim}_{(U,u)\in\mathscr{U}_0(s)^{opp}}\, \mathscr{O}_U(U).$$

Let $s \in S$ and $\bar{s} \xrightarrow{i} S$ be a geometric point of S with image s.

(a) Show that we obtain natural ring homomorhisms

$$\mathscr{O}_{S,s} \to \mathscr{O}^h_{S,s} \to \mathscr{O}_{S,\bar{s}}.$$

(b) Show that $\mathscr{O}^h_{S,s}$ is a local ring with maximal ideal $\mathfrak{m}_s\mathscr{O}^h_{S,s}$ and residue field $\kappa(s)$.

(c) Show that $\mathscr{O}^h_{S,s}$ is a henselian local ring.

(d) Show that $\mathscr{O}_{S,\bar{s}}$ is a strictly henselian local ring with maximal ideal $\mathfrak{m}_s\mathscr{O}_{S,\bar{s}}$. Moreover, show that the separabel closure of $\kappa(s)$ in $\kappa(\bar{s})$ is the associated residue field.

(e) Show that the homomorphisms from (a) are local and faithfully flat.

Remarks: To show these assertions, one could for example proceed as follows.
To (b):

(i) Use the following lemma (without a proof) to reduce to the case $S = \text{Spec}(R)$ is the spectrum of a local ring $R = (R, \mathfrak{m}, \kappa)$ and s corresponds to $\mathfrak{m}$:

Lemma 1. *(c.f. Stacks Project Tag 04GV) Let A be a ring and denote its affine spectrum* $\mathrm{Spec}(A)$ *by X. Let $x \in X$ with associated prime ideal $\mathfrak{p} \subseteq A$. Consider the category of pairs $(S, \mathfrak{q})$ where* $\mathrm{Spec}(S) \to \mathrm{Spec}(A_\mathfrak{p})$ *is an étale* $\mathrm{Spec}(A)$*-scheme and $\mathfrak{q} \subseteq S$ is a prime lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. This category is filtered and we have the following natural isomorphism*

$$\mathscr{O}^h_{X,x} \cong \mathrm{colim}_{(S,\mathfrak{q})} S.$$

(ii) Show that for every étale morphism $g\colon \mathrm{Spec}(A) \to \mathrm{Spec}(R) = S$ mapping $\mathfrak{p} \in \mathrm{Spec}(A)$ to $\mathfrak{m}$, the maximal ideal of R, there exists an open affine subscheme $\mathrm{Spec}(A') \subseteq \mathrm{Spec}(A)$ such that $\mathrm{Spec}(A')$ contains $\mathfrak{p}$ and $\mathfrak{p}$ is the only prime ideal of A' lying over $\mathfrak{m}$. Moreover, show that in this case we have $\mathfrak{m}A' = \mathfrak{p}$ and $A'/\mathfrak{m}A' = \kappa$.

(iii) Show that every element of $R^h \setminus \mathfrak{m}R^h$ is invertible in R^h.

(iv) Show that $R^h/\mathfrak{m}R^h = \kappa$.

To (c):

(i) Take a monic polynomial $P \in R^h[T]$ such that its reduction $\overline{P} \in \kappa[T]$ has 0 as a simple root. Show that we can find an object $(\mathrm{Spec}(A), \mathfrak{q})$ in $\mathscr{U}_0(s)$ such that there exists a monic poynomial $Q \in A[T]$ that is mapped to P under $A[T] \to R^h[T]$.

(ii) We set $A' = A[T]/(Q)$ and $\mathfrak{q}' = (\mathfrak{q}, T)$ to be the ideal in A' generated by $\mathfrak{q}A'$ and T. Show that $\mathfrak{q}' \in \mathrm{Spec}(A')$ and $\kappa(\mathfrak{q}') = \kappa$.

(iii) Use Exercise H4 to show that there is an open affine neighbourhood $\mathrm{Spec}(A'') \subseteq \mathrm{Spec}(A')$ that contains $\mathfrak{q}'$ and which defines an object $(\mathrm{Spec}(A''), \mathfrak{q}'A'')$ of $\mathscr{U}_0(s)$.

(iv) Find an element of A'' which is mapped to a root of P under the map $A'' \to R^h$.

To (d): Adapt the arguments from (b) and (c). There is also a lemma in the style of Lemma 1 (c.f. Stacks Project Tag 04GW) which shows that we can assume $S = \mathrm{Spec}(R)$ for a local ring R in this case as well.

Exercise H35 (Contractible objects) (6+6 points)

(a) Let G be a group and let $\mathscr{C} := (G\text{-Sets})$ denote the site of left G-sets (c.f. Exercise H13). Let ${}_GG$ denote G considered as an object of $\mathscr{C}$. Show that $\mathscr{B} := \{{}_GG\} \subseteq \mathrm{Ob}(\mathscr{C})$ satisfies the properties (a) and (b) from Proposition 6.29.

(b) Find examples of w-contractible rings.