(Pro-) Étale Cohomology
10. Exercise Sheet

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Homework

Exercise H30 (6+6 points)

Let X be a noetherian integral scheme with generic point η and function field $K := \mathcal{O}_{X,\eta}$. Let $j\colon \operatorname{Spec}(K) \to X$ be the canonical morphism and denote the induced morphism of sites $\operatorname{Spec}(K)_{\text{ét}} \to X_{\text{ét}}$ also by j. Recall that j is given by a continuous functor $\varphi\colon X_{\text{ét}} \to \operatorname{Spec}(K)_{\text{ét}}, U \mapsto U \times_X \operatorname{Spec}(K)$ which induces a functor $j_* = \varphi^s\colon \operatorname{Ab}(\operatorname{Spec}(K)_{\text{ét}}) \to \operatorname{Ab}(X_{\text{ét}})$. Let $(\mathbb{G}_m)_X$ (resp. $(\mathbb{G}_m)_K := (\mathbb{G}_m)_{\operatorname{Spec}(K)}$) denote the sheaf of abelian groups $T \mapsto \Gamma(T, \mathcal{O}_T)$ on $X_{\text{ét}}$ (resp. on $\operatorname{Spec}(K)_{\text{ét}}$). We obtain a morphism of abelian sheaves on $X_{\text{ét}}$:

$$\nu\colon (\mathbb{G}_m)_X \to j_*((\mathbb{G}_m)_K)$$

(a) Show that ν is injective.

We define the sheaf of abelian groups $\operatorname{Div}_X := \operatorname{Div}_{X_{\text{ét}}}$ on $X_{\text{ét}}$ to be the cokernel of ν. We call Div_X the sheaf of *(Cartier) divisors on $X_{ét}$* and obtain a short exact sequence

$$0 \to (\mathbb{G}_m)_X \xrightarrow{\nu} j_*((\mathbb{G}_m)_K) \to \operatorname{Div}_X \to 0.$$

(b) Show that the global sections $\operatorname{Div}_X(X)$ coincide with the set of Cartier divisors $\operatorname{Div}(X) = \operatorname{Div}_{X_{\text{Zar}}}(X)$ on X_{Zar} (here $\operatorname{Div}_{X_{\text{Zar}}}$ is defined as above but using the respective Zariski sites, c.f. [1] §11.11).

Let $X^{(1)} := \{x \in X \mid \dim \mathcal{O}_{X,x} = 1\}$. For all $x \in X^{(1)}$ let $i_x\colon \operatorname{Spec}(\kappa(x)) \to X$ be the canonical inclusion and let $\mathbb{Z}_x$ denote the constant sheaf on $\operatorname{Spec}(\kappa(x))_{\text{ét}}$ with values in $\mathbb{Z}$. As before, the associated morphism of sites $(\operatorname{Spec}(\kappa(x)))_{\text{ét}} \to X_{\text{ét}}$ is also denoted by i_x. There is a canonical morphism of sheaves on $X_{\text{ét}}$

$$\operatorname{cyc}\colon \operatorname{Div}_X \to \bigoplus_{x \in X^{(1)}} (i_x)_*(\mathbb{Z}_x),$$

where the global sections of the right hand side can be identified with the abelian group of Weil divisors on X (c.f. [1] §11.13). If X is locally factorial, one can show that cyc is an isomorphism (c.f. [1] Theorem 11.38(2)). In particular, if X is regular (which implies that $\mathcal{O}_{X,x}$ is a discrete valuation ring for all $x \in X^{(1)}$), cyc is an isomorphism by the theorem of Auslander-Buchsbaum (c.f. [1] B.74).

Exercise H31 (6+6 points)

(a) Let G be a profinite group and A be a torsion free abelian group which we consider as a G-module via the trivial G-action on A. Show that

$$H^1(G,A) = 0.$$

We take the notation from Exercise H30 and assume X to be regular.

(b) Show that

$$H^1_{\text{ét}}(X, \operatorname{Div}_X) \cong \bigoplus_{x \in X^{(1)}} H^1_{\text{ét}}(X, (i_x)_*(\mathbb{Z}_x)) = 0.$$

Remark: It might be useful to consider the Leray spectral sequence (c.f. Corollary 4.16) for the canonical morphism of sites $i\colon (\operatorname{Spec}(\kappa(x))_{\text{ét}} \to X_{\text{ét}}$ and $\mathcal{F} = \mathbb{Z}_x \in \operatorname{Ab}(\operatorname{Spec}(\kappa(x))_{\text{ét}})$. Then one could for example look at the associated low terms exact sequence (c.f. 4.10).

Exercise H32 (6+3+3 points)

(a) Let X be a scheme, $x \in X$ and $i\colon \mathrm{Spec}(\kappa(x)) \to X$ denote the canonical morphism. The associated morphism of sites $\mathrm{Spec}(\kappa(x))_{\text{ét}} \to X_{\text{ét}}$ is also denoted by i. Show that $R^1 i_*((\mathbb{G}_m)_{\kappa(x)}) = 0$. Moreover, conclude that the edge morphism from the Leray spectral sequence associated with i and $(\mathbb{G}_m)_{\kappa(x)}$ (c.f. (4.10))

$$H^2_{\text{ét}}(X, i_*((\mathbb{G}_m)_{\kappa(x)}) \to H^2_{\text{ét}}(\mathrm{Spec}(\kappa(x)), (\mathbb{G}_m)_{\kappa(x)})$$

is injective.

(b) Let X be an integral regular noetherian scheme with function field K. Show that there is a canonical injection

$$H^2_{\text{ét}}(X, (\mathbb{G}_m)_X) \hookrightarrow Br(K).$$

(c) Let k be an algebraically closed field and X be a regular curve over k (i.e. X is a regular integral scheme of dimension 1 which is of finite type over k). Show that for all $n \in \mathbb{N}$ which are prime to the characteristic of k we have an exact sequence

$$0 \to H^0_{\text{ét}}(X, (\mu_n)_X) \to H^0_{\text{ét}}(X, \mathcal{O}_X^*) \xrightarrow{(\cdot)^n} H^0_{\text{ét}}(X, \mathcal{O}_X^*) \to H^1_{\text{ét}}(X, (\mu_n)_X) \to \mathrm{Pic}(X) \xrightarrow{\cdot n} \mathrm{Pic}(X) \to H^2_{\text{ét}}(X, (\mu_n)_X) \to 0.$$

References

[1] U. Görtz, T. Wedhorn. *Algebraic Geometry I*. Vieweg und Teubner, 2010