ERRATUM TO “AFFINE GRASSMANNIANS AND GEOMETRIC SATAKE EQUIVALENCE’

The references below refer to the authors work [Ri16]. We address Lemma 2.17, Theorem 2.19 and Definition 3.3.

0.1. On Lemma 2.17. This lemma is wrong, and should be replaced by Lemma 0.1 below. It is only used in the proof of Lemma 2.20 to show the ind-finiteness of

\[(0.1) \text{Gr}(\text{Res}_{\tilde{X}/X}(\mathbb{G}_m), X) \to X,\]

where \(\tilde{X} \to X\) is a finite, flat surjection of quasi-compact, separated, smooth curves over \(k\). Since (0.1) is ind-quasi-finite, it is enough to show its ind-properness. More generally, let \(\tilde{G} \to \tilde{X}\) be any smooth, \(\tilde{X}\)-affine group scheme. We aim to relate the Beilinson-Drinfeld Grassmannian for \(\text{Res}_{\tilde{X}/X}(\tilde{G})\) to the general set-up introduced in [HR, §3] which is recalled in (0.2) below for convenience. Then the ind-properness of (0.1) follows from the comparison in Lemma 0.1 below, and the representability result in [HR, Lem. 3.7] applied with \(\tilde{G} = \text{GL}_n, \tilde{X}, n = 1\). More generally, the ind-properness holds true whenever \(\tilde{G} = G_0 \times_k \tilde{X}\) where \(G_0\) is a reductive \(k\)-group, cf. [HR, Cor. 3.10 iii)].

Recall the set-up in [HR, §3]. Let \(S\) be a Noetherian scheme, and let \(Y \to S\) be a scheme which is quasi-compact, separated and smooth of pure relative dimension 1. Let \(D \subset Y\) be a relative effective Cartier divisor which is finite and flat over \(S\). Let \(\tilde{G} \to Y\) be a smooth, \(Y\)-affine group scheme. To \((Y, \tilde{G}, D)\) we associate the functor \(\text{Gr}_{(Y, \tilde{G}, D)}\) on the category of \(S\)-schemes which assigns to every \(T \to S\) the set of isomorphism classes of pairs \((E, \alpha)\) with

\[(0.2) \begin{cases} E \text{ a } \tilde{G}\text{-torsor on } Y \times_S T, \\ \alpha : E|_{(Y \setminus D) \times_S T} \xrightarrow{\cong} E_{0}|_{(Y \setminus D) \times_S T} \text{ a trivialization}, \end{cases} \]

where \(E_0\) denotes the trivial torsor.

Lemma 0.1. Let \(\tilde{X} \to X\) be a finite, flat surjection of quasi-compact, separated, smooth curves over \(k\), and let \(\tilde{G} \to \tilde{X}\) be a smooth, \(\tilde{X}\)-affine group scheme. Then restriction of scalars along \(\tilde{X} \to X\) induces an isomorphism of functors over \(X\),

\[\text{Gr}_{(Y, \tilde{G}, D)} \xrightarrow{\cong} \text{Gr}(\text{Res}_{\tilde{X}/X}(\tilde{G}), X),\]

where \(Y := \tilde{X} \times_k X \xrightarrow{p_X} X =: S\) is viewed as a relative curve, \(\tilde{G} := \tilde{G} \times \tilde{X} Y\) and \(D := \tilde{X} \to Y\) is viewed as a relative effective Cartier divisor over \(S\).

Proof. Let \(T \to S\) be a scheme, and let \((E, \alpha) \in \text{Gr}_{(Y, \tilde{G}, D)}(T)\). By definition, \(E\) is a \(\tilde{G}\)-torsor on \(Y \times_X T = \tilde{X} \times_k T\) together with a trivialization \(\alpha\) on \((Y \setminus D) \times_S T = \tilde{X} \times_k T \setminus \Gamma\) where \(\Gamma\) is defined by the Cartesian diagram

\[
\begin{array}{ccc}
\Gamma & \longrightarrow & \tilde{X} \times_k T =: \tilde{X}_T \\
\downarrow & & \downarrow \\
T & \longrightarrow & X \times_k T =: X_T.
\end{array}
\]

Here the bottom arrow is the graph of \(T \to X\) which is a closed immersion because \(X\) is separated over \(k\). Thus, to \((E, \alpha)\) we can associate the pair \((E', \alpha') \in \text{Gr}(\text{Res}_{\tilde{X}/X}(\tilde{G}), X)(T),\)
where $\mathcal{E}' := \text{Res}_{\bar{X}/X}(\mathcal{E})$ and $\alpha'$ is the trivialization corresponding to $\alpha$ under

$$\text{Isom}(\mathcal{E}, \mathcal{E}_0)(\bar{X}/\Gamma) = \text{Isom}(\mathcal{E}', \mathcal{E}_0')(X/T).$$

This gives the desired isomorphism, cf. [Lev13, Thm. 2.6.1] for details. □

The problem with the proof of Lemma 2.17 is that $\text{Res}_{\bar{X}/X}(\text{Gr}(\hat{G}, \bar{X}))(T)$ parametrizes torsors on $\bar{X} \times_k (\bar{X} \times X T)$ which is obviously different from $\bar{X} \times_k T$.

0.2. On Theorem 2.19. João Lourenço observed that the statement of Theorem 2.19 is missing a hypothesis if $\text{char}(k) > 0$. Namely, one has to add “Assume that the generic fibre of $G$ is reductive.”. As argued above, Lemma 0.1 applied with $k = \mathbb{F}_p$, $\bar{X} := \mathbb{A}_k^1 \to (\mathbb{A}_k^1)^{(p)} =: X$ the relative Frobenius, and $\hat{G} = G_0 \times_k \bar{X}$ with $G_0$ any reductive $k$-group gives examples where

$$\text{Gr}(\text{Res}_{\bar{X}/X}(\hat{G}), X) \to X$$

is ind-proper, but the generic fibre of $\text{Res}_{\bar{X}/X}(\hat{G}) \to X$ is not reductive.

In the proof of Theorem 2.19, the following sentence on middle of page 3731 is wrong: “But $\text{Gr}(G, X)_{\bar{q}}$ is the affine Grassmannian associated with the linear algebraic group $G_{\bar{q}} [\ldots]$.” By Corollary 2.14, the fibre $\text{Gr}(G, X)_{\bar{q}}$ is canonically the affine Grassmannian associated with the group scheme $G \otimes_X F[t_{\bar{q}}]$ where $\bar{q} = \text{Spec}(F)$, and $\text{Spec}(F[t_{\bar{q}}]) \to X \times_k \bar{q} \to \bar{q}$ is the canonical map given by completion along the tautological point on the curve $X \times_k \bar{q} \to \bar{q}$ with local parameter $t_{\bar{q}}$. However, in general

$$G \otimes_X F[t_{\bar{q}}] \neq G_{\bar{q}} \otimes_{\bar{q}} F[t_{\bar{q}}],$$

where $\text{Spec}(F[t_{\bar{q}}]) \to X \times_k \bar{q} \to \bar{q}$ is the canonical map. In other words, $G \otimes_X F[t_{\bar{q}}]$ might not be a constant family. The following lemma shows that this phenomenon does not arise when the generic fibre of $G \to X$ is reductive:

**Lemma 0.2.** Let $(A, m)$ be a Henselian local ring with residue field $\kappa := A/m$. Assume that there exists a section $\kappa \to A$, i.e., $A$ is a $\kappa$-algebra, and the composition $\kappa \to A \to \kappa$ is the identity. Then any (fibrewise connected) reductive group scheme $G$ over $A$ is constant, i.e., there exists an isomorphism of $A$-algebra schemes

$$G \simeq (G \otimes_A \kappa) \otimes_{\kappa} A.$$

**Remark 0.3.** We apply the lemma to the pair $(A, m) = (F[t], (t))$, and the reductive group scheme $G = G \otimes_X F[t]$ where $t = t_{\bar{q}}$ is a local coordinate of the tautological point of $X \times_k \bar{q} \to \bar{q}$, $\bar{q} = \text{Spec}(F)$. Since in [Ri16, §3ff.] the generic fibre of $G \to X$ is assumed to be reductive, the rest of the manuscript is unaffected from the mistake.

**Proof.** As any reductive group scheme splits étale locally, and as $A$ is Henselian, we claim that there exists a finite Galois extension $\kappa'/\kappa$ such that $G$ splits over $A' := A \otimes_{\kappa} \kappa'$, i.e., one has $G \otimes_A A' \simeq G_0 \otimes_{\kappa} A'$ for some Chevalley group $G_0$ over $\kappa$. Indeed, take a finite type étale cover $A \to B$ which splits $G$, cf. [Co14, Lem. 5.1.3]. As $A$ is Henselian, the $A$-algebra $B$ is a finite direct product of finite local $A$-algebras which are necessarily étale, cf. [StaPro, 03QH]. Let $A'$ be one of the factors. By [StaPro, 09ZS] and the existence of $\kappa \to A$, the $A$-algebra $A'$ is necessarily of the form $A' \simeq A \otimes_{\kappa} \kappa'$ where $\kappa'$ is the residue field of $A'$. Enlarging $\kappa'$ if necessary, we may assume that $\kappa'/\kappa$ is a finite Galois extension. The existence of $G_0$ follows from the Isomorphism Theorem [Co14, Thm. 6.1.17]. This shows the claim. Now the isomorphism class of $G$ corresponds to a class in the cohomology set

$$H^1(A'/A, \text{Aut}(G_0) \otimes_{\kappa} A).$$

We have $\text{Aut}(G_0) = G_{0, \text{ad}} \rtimes \text{Aut}(R, \Delta)_{\kappa}$ where $\text{Aut}(R, \Delta)$ is the abstract group of automorphisms of some based root datum of $(R, \Delta)$ of $G_0$ (cf. [Co14, Thm. 7.1.9]). Since $A'/A$ is a finite Galois cover with group $\Gamma := \text{Aut}(\kappa'/\kappa)$, we have

$$H^1(A'/A, \text{Aut}(R, \Delta)_{A}) = \text{Hom}_{\text{Grps}}(\Gamma, \text{Aut}(R, \Delta)) = H^1(\kappa'/\kappa, \text{Aut}(R, \Delta)_{\kappa}).$$
i.e., the quasi-split form $G^*$ of $G$ is defined over $\kappa$. Now the isomorphism class of $G$ corresponds to a class in the cohomology set

$$H^1(A'/A, G^*_{ad} \otimes_{\kappa} A).$$

We claim that for any smooth affine $\kappa$-group $H$, the natural reduction map

$$(0.3) \quad H^1(A'/A, H \otimes_{\kappa} A) \to H^1(\kappa'/\kappa, H)$$

is bijective. This claim applied to $H = G^*_{ad}$ implies the lemma. To prove the claim we observe that the map is surjective because any torsor on $\text{Spec}(\kappa)$ can be extended constantly to a torsor on $\text{Spec}(A)$ using the section $\kappa \to A$. To show that $(0.3)$ is injective, it suffices (by the twisting trick) to prove the triviality of the kernel. Let $E \to \text{Spec}(A)$ be an $H$-torsor, and suppose $E|_{\kappa}$ is trivial, equivalently $E(\kappa)$ is non-empty. As $H$ is a smooth, affine $\kappa$-group scheme, the torsor $E \to \text{Spec}(A)$ is representable by a smooth affine scheme which follows from effectivity of descent for affine maps, cf. [StaPro, 0246, 02VL]. Hence, the reduction map $E(A) \to E(\kappa)$ is surjective because $A$ is Henselian, cf. [EGA IV$_4$, Thm. 18.5.17]. Thus, $E(A)$ is non-empty, or equivalently $E$ is trivial. This proves the claim.

On the positive side, we have the following result. Let $G$ be a smooth, affine $F$-group scheme. We denote by $\text{Gr}_G := \text{Gr}(G, X)_\eta$ the generic fibre where $G \to X$ denotes a smooth, affine model of $G$ (this exists after possibly shrinking $X$). Then $\text{Gr}_G$ is well-defined up to isomorphism independently of the choice of the model $(G, X)$, and representable by a separated $F$-ind-scheme of ind-finite type.

Recall from [CGP10, Def. B.2.1] that a smooth, connected, unipotent $F$-group $U$ is called $F$-wound if every map of $F$-schemes $A^* \to U$ is a constant map to a point in $U(F)$.

**Lemma 0.4.** Let $G$ be a smooth, affine $F$-group. If $\text{Gr}_G \to \text{Spec}(F)$ is ind-proper, then the neutral component $G^0$ is quasi-reductive in the sense of [BT84, 1.1.12], i.e., the unipotent radical $U := R_u(G^0)$ is $F$-wound.

**Remark 0.5.** Lemma 0.4 shows that for characteristic zero fields $F$ the unipotent radical $R_u(G^0)$ is trivial, so that $G^0$ is indeed reductive. If $F$ is of positive characteristic (and hence non-perfect), then by Example 0.6 below ind-properness of $\text{Gr}_G \to \text{Spec}(F)$ does not imply that $G^0$ is pseudo-reductive in the sense of [CGP10, Def. 1.1.1], i.e., the unipotent radical $U$ needs not to be trivial. It would be interesting to see whether the converse of Lemma 0.4 holds, i.e., whether quasi-reductive $F$-groups $G$ are characterized by the property that $\text{Gr}_G \to \text{Spec}(F)$ is ind-proper.

**Proof.** We use the following principle: If $H \subset G$ is an $F$-smooth, closed, normal subgroup, and if $\text{Gr}_G \to \text{Spec}(F)$ is ind-proper, then $\text{Gr}_H \to \text{Spec}(F)$ is ind-proper as well. Indeed, the fpf (or étale) quotient $G/H$ is an affine scheme, and therefore $\text{Gr}_H \to \text{Gr}_G$ is representable by a closed immersion, cf. [HR, Prop. 3.9]. The principle shows that if $\text{Gr}_G \to \text{Spec}(F)$ is ind-proper, then $\text{Gr}_U \to \text{Spec}(F)$ is ind-proper as well. Now by Tits’ structure theory for unipotent groups [CGP10, App. B] there exists a short exact sequence $1 \to U_s \to U \to U/U_s \to 1$ where $U_s$ is an $F$-split, connected, unipotent, normal subgroup of $U$, and the quotient $U/U_s$ is an $F$-wound, unipotent group, cf. [CGP10, Thm. B.3.4]. Note that $\text{Gr}_{G_{a,F}} \simeq \colim_{n \geq 0} A^n_F$ is an infinite-dimensional affine space, so in particular not ind-proper. Hence, if $\text{Gr}_U \to \text{Spec}(F)$ is ind-proper, then our principle implies that $U_s$ is trivial, so that $U$ is $F$-wound.

**Example 0.6.** The following example is due to J. Lourenço. Let $F = \mathbb{F}_p(t)$ and consider the inseparable field extension $\tilde{F} = \mathbb{F}_p(t^{1/2})$. Define the smooth, affine, connected $F$-group $G$ by the exact sequence

$$1 \to \mathbb{G}_{m,F} \to \text{Res}_{\tilde{F}/F}(\mathbb{G}_{m,\tilde{F}}) \to G \to 1.$$
Then $G$ is commutative, unipotent of dimension $p - 1 > 0$ (cf. [CGP10, Exam. 1.1.3]), and therefore not pseudo-reductive. However, since $H^1_{et}(K((z)), \mathbb{G}_m)$ vanishes for any field extension $K/F$, it is easy to see that the map of ind-schemes

$$\text{Gr}_{\text{Res}_F/F}(G_m, F) \to \text{Gr}_G$$

is surjective. As the source of this map is ind-proper over $F$ and the target is separated, we see that $\text{Gr}_G \to \text{Spec}(F)$ is ind-proper as well.

0.3. On Definition 3.3. In this definition, the Beilinson-Drinfeld affine Grassmannian $\text{Gr}_G$ is defined for any smooth, affine group scheme $\mathcal{G}$ over $k[t]$ with (connected) reductive generic fibre. The definition uses a spreading $\mathcal{G}_X$ of $\mathcal{G}$ over some smooth, affine, pointed $k$-curve $(X, x)$, i.e., one has $\mathcal{O}_{X,x} \simeq k[t]$ on completed local rings, and $\mathcal{G}_X \to X$ is a smooth, $X$-affine group scheme with $\mathcal{G}_X \otimes X \mathcal{O}_{X,x} \simeq \mathcal{G}$. A direct way of defining $\text{Gr}_G \to \text{Spec}(k[t])$ without using a spreading $\mathcal{G}_X$ is as follows.

We define $\text{Gr}_G$ to be the functor on the category of $k[t]$-algebras $R$ given by the isomorphism classes of tuples $(\mathcal{E}, \alpha)$ with

$$
(0.4) \quad \begin{cases}
\mathcal{E} \text{ a } \mathcal{G} \otimes_{k[t]} R[z - t]\text{-torsor on } \text{Spec}(R[z - t]), \\
\alpha : \mathcal{E}|_{\text{Spec}(R[z - t])} \simeq \mathcal{E}_0|_{\text{Spec}(R[z - t])} \text{ a trivialization,}
\end{cases}
$$

where $\mathcal{E}_0$ denotes the trivial torsor. Here $z$ is an additional formal variable, and the map $k[t] \to R[z - t]$ is the unique $k$-algebra map with the property $t \mapsto z$: the existence of this map is verified by writing a power series in $z$ as

$$
\sum_{i \geq 0} a_i z^i = \sum_{i \geq 0} b_i (z - t)^i \in R[z - t],
$$

where $a_i \in k$ are some coefficients, and $b_i = b_i(\{a_i\}, t) \in R$ are power series in $t$ determined by the equation. The functor (0.4) agrees with Definition 3.3 defined using a spreading of $\mathcal{G}$ over some curve, and is therefore representable by an ind-projective ind-scheme over $k[t]$. The generic fibre of $\text{Gr}_G$ is canonically the affine Grassmannian for the reductive group scheme $\mathcal{G} \otimes_{k[t]} F[z - t] \simeq G \otimes_F F[z - t]$ (cf. Lemma 0.2) where $F := k((t))$ and $G := \mathcal{G} \otimes_{k[t]} F$. Its special fibre is canonically the twisted affine flag variety for $\mathcal{G}$ over $k[t]$ in the sense of [PR08].

References


[Ri16] T. Richarz: Affine Grassmannians and geometric Satake equivalences, IMRN 12 (2016), 3717-3767. 1, 2


Technical University of Darmstadt, Department of Mathematics, 64289 Darmstadt, Germany

Email address: richarz@mathematik.tu-darmstadt.de