ERRATUM TO “AFFINE GRASSMANNIANS AND GEOMETRIC SATAKE EQUIVALENCES’

The references below refer to the authors work [Ri16]. We adress Lemma 2.17, Theorem 2.19 and Definition 3.3.

0.1. **On Lemma 2.17.** This lemma is wrong, and should be replaced by Lemma 0.1 below. It is only used in the proof of Lemma 2.20 to show the ind-finiteness of

$$\mathrm{Gr}\big(\mathrm{Res}_{\tilde{X}/X}(\mathbb{G}_m), X\big) \to X, \tag{0.1}$$

where $\tilde{X} \to X$ is a finite, flat surjection of quasi-compact, separated, smooth curves over k. Since (0.1) is ind-quasi-finite, it is enough to show its ind-properness. More generally, let $\tilde{\mathcal{G}} \to \tilde{X}$ be any smooth, $\tilde{X}$-affine group scheme. We aim to relate the Beilinson-Drinfeld Grassmannian for $\mathrm{Res}_{\tilde{X}/X}(\tilde{\mathcal{G}})$ to the general set-up introduced in [HR, §3] which is recalled in (0.2) below for convenience. Then the ind-properness of (0.1) follows from the comparison in Lemma 0.1 below, and the representability result in [HR, Lem. 3.7] applied with $\tilde{\mathcal{G}} = \mathrm{GL}_{n,\tilde{X}}$, $n = 1$. More generally, the ind-properness holds true whenever $\tilde{\mathcal{G}} = G_0 \times_k \tilde{X}$ where G_0 is a reductive k-group, cf. [HR, Cor. 3.10 iii)].

Recall the set-up in [HR, §3]. Let S be a Noetherian scheme, and let $Y \to S$ be a scheme which is quasi-compact, separated and smooth of pure relative dimension 1. Let $D \subset Y$ be a relative effective Cartier divisor which is finite and flat over S. Let $\mathcal{G} \to Y$ be a smooth, Y-affine group scheme. To $(Y, \mathcal{G}, D)$ we associate the functor $\mathrm{Gr}_{(Y,\mathcal{G},D)}$ on the category of S-schemes which assigns to every $T \to S$ the set of isomorphism classes of pairs $(\mathcal{E}, \alpha)$ with

$$\begin{cases} \mathcal{E} \text{ a } \mathcal{G}\text{-torsor on } Y \times_S T, \\ \alpha : \mathcal{E}|_{(Y\backslash D)\times_S T} \xrightarrow{\simeq} \mathcal{E}_0|_{(Y\backslash D)\times_S T} \text{ a trivialization,} \end{cases} \tag{0.2}$$

where $\mathcal{E}_0$ denotes the trivial torsor.

Lemma 0.1. *Let $\tilde{X} \to X$ be a finite, flat surjection of quasi-compact, separated, smooth curves over k, and let $\tilde{\mathcal{G}} \to \tilde{X}$ be a smooth, $\tilde{X}$-affine group scheme. Then restriction of scalars along $\tilde{X} \to X$ induces an isomorphism of functors over X,*

$$\mathrm{Gr}_{(Y,\mathcal{G},D)} \xrightarrow{\simeq} \mathrm{Gr}\big(\mathrm{Res}_{\tilde{X}/X}(\tilde{\mathcal{G}}), X\big),$$

where $Y := \tilde{X} \times_k X \overset{\mathrm{pr}}{\to} X =: S$ is viewed as a relative curve, $\mathcal{G} := \tilde{\mathcal{G}} \times_{\tilde{X}} Y$ and $D := \tilde{X} \hookrightarrow Y$ is viewed as a relative effective Cartier divisor over S.

Proof. Let $T \to S$ be a scheme, and let $(\mathcal{E}, \alpha) \in \mathrm{Gr}_{(Y,\mathcal{G},D)}(T)$. By definition, $\mathcal{E}$ is a $\mathcal{G}$-torsor on $Y \times_X T = \tilde{X} \times_k T$ together with a trivialization α on $(Y\backslash D) \times_S T = \tilde{X} \times_k T\backslash\Gamma$ where Γ is defined by the Cartesian diagram

$$\begin{array}{ccc} \Gamma & \longrightarrow & \tilde{X} \times_k T =: \tilde{X}_T \\ \downarrow & & \downarrow \\ T & \longrightarrow & X \times_k T =: X_T. \end{array}$$

Here the bottom arrow is the graph of $T \to X$ which is a closed immersion because X is separated over k. Thus, to $(\mathcal{E}, \alpha)$ we can associate the pair

$$(\mathcal{E}', \alpha') \in \mathrm{Gr}\big(\mathrm{Res}_{\tilde{X}/X}(\tilde{\mathcal{G}}), X\big)(T),$$

where $\mathcal{E}' := \mathrm{Res}_{\tilde{X}_T/X_T}(\mathcal{E})$ and α' is the trivialization corresponding to α under

$$\mathrm{Isom}(\mathcal{E},\mathcal{E}_0)(\tilde{X}_T\backslash\Gamma) \;=\; \mathrm{Isom}(\mathcal{E}',\mathcal{E}_0')(X_T\backslash T).$$

This gives the desired isomorphism, cf. [Lev13, Thm. 2.6.1] for details. □

The problem with the proof of Lemma 2.17 is that $\mathrm{Res}_{\tilde{X}/X}\big(\mathrm{Gr}(\tilde{\mathcal{G}},\tilde{X})\big)(T)$ parametrizes torsors on $\tilde{X}\times_k(\tilde{X}\times_X T)$ which is obviously different from $\tilde{X}\times_k T$.

0.2. **On Theorem 2.19.** João Lourenço observed that the statement of Theorem 2.19 is missing a hypothesis if $\mathrm{char}(k)>0$. Namely, one has to add "*Assume that the generic fibre of $\mathcal{G}$ is reductive.*". As argued above, Lemma 0.1 applied with $k=\mathbb{F}_p$, $\tilde{X}:=\mathbb{A}^1_k\to(\mathbb{A}^1_k)^{(p)}=:X$ the relative Frobenius, and $\tilde{\mathcal{G}}=G_0\times_k\tilde{X}$ with G_0 any reductive k-group gives examples where

$$\mathrm{Gr}(\mathrm{Res}_{\tilde{X}/X}(\tilde{\mathcal{G}}),X)\to X$$

is ind-proper, but the generic fibre of $\mathrm{Res}_{\tilde{X}/X}(\tilde{\mathcal{G}})\to X$ is not reductive.

In the proof of Theorem 2.19, the following sentence on middle of page 3731 is wrong: "*But $\mathrm{Gr}(\mathcal{G},X)_{\bar\eta}$ is the affine Grassmannian associated with the linear algebraic group $\mathcal{G}_{\bar\eta}$ [...]*.". By Corollary 2.14, the fibre $\mathrm{Gr}(\mathcal{G},X)_{\bar\eta}$ is canonically the affine Grassmannian associated with the group scheme $\mathcal{G}\otimes_X\bar{F}[\![t_{\bar\eta}]\!]$ where $\bar\eta=\mathrm{Spec}(\bar{F})$, and $\mathrm{Spec}(\bar{F}[\![t_{\bar\eta}]\!])\to X\times_k\bar\eta\to X$ is the canonical map given by completion along the tautological point on the curve $X\times_k\bar\eta\to\bar\eta$ with local parameter $t_{\bar\eta}$. However, in general

$$\mathcal{G}\otimes_X\bar{F}[\![t_{\bar\eta}]\!]\;\not\simeq\;\mathcal{G}_{\bar\eta}\otimes_{\bar\eta}\bar{F}[\![t_{\bar\eta}]\!],$$

where $\mathrm{Spec}(\bar{F}[\![t_{\bar\eta}]\!])\to X\times_k\bar\eta\to\bar\eta$ is the canonical map. In other words, $\mathcal{G}\otimes_X\bar{F}[\![t_{\bar\eta}]\!]$ might not be a constant family. The following lemma shows that this phenomenon does not arise when the generic fibre of $\mathcal{G}\to X$ is reductive:

Lemma 0.2. *Let $(A,\mathfrak{m})$ be a Henselian local ring with residue field $\kappa:=A/\mathfrak{m}$. Assume that there exists a section $\kappa\to A$, i.e., A is a κ-algebra, and the composition $\kappa\to A\to\kappa$ is the identity. Then any (fibrewise connected) reductive group scheme G over A is constant, i.e., there exists an isomorphism of A-group schemes*

$$G\;\simeq\;\big(G\otimes_A\kappa\big)\otimes_\kappa A.$$

Remark 0.3. We apply the lemma to the pair $(A,\mathfrak{m})=(F[\![t]\!],(t))$, and the reductive group scheme $G=\mathcal{G}\otimes_X F[\![t]\!]$ where $t=t_\eta$ is a local coordinate of the tautological point of $X\times_k\eta\to\eta$, $\eta=\mathrm{Spec}(F)$. Since in [Ri16, §3ff.] the generic fibre of $\mathcal{G}\to X$ is assumed to be reductive, the rest of the manuscript is unaffected from the mistake.

Proof. As any reductive group scheme splits étale locally, and as A is Henselian, we claim that there exists a finite Galois extension κ'/κ such that G splits over $A':=A\otimes_\kappa\kappa'$, i.e., one has $G\otimes_A A'\simeq G_0\otimes_\kappa A'$ for some Chevalley group G_0 over κ. Indeed, take a finite type étale cover $A\to B$ which splits G, cf. [Co14, Lem. 5.1.3]. As A is Henselian, the A-algebra B is a finite direct product of finite local A-algebras which are necessarily étale, cf. [StaPro, 03QH]. Let A' be one of the factors. By [StaPro, 09ZS] and the existence of $\kappa\to A$, the A-algebra A' is necessarily of the form $A'\simeq A\otimes_\kappa\kappa'$ where κ' is the residue field of A'. Enlarging κ' if necessary, we may assume that κ'/κ is a finite Galois extension. The existence of G_0 follows from the Isomorphism Theorem [Co14, Thm. 6.1.17]. This shows the claim. Now the isomorphism class of G corresponds to a class in the cohomology set

$$H^1\big(A'/A,\;\underline{\mathrm{Aut}}(G_0)\otimes_\kappa A\big).$$

We have $\underline{\mathrm{Aut}}(G_0)=G_{0,\mathrm{ad}}\rtimes\mathrm{Aut}(R,\Delta)_\kappa$ where $\mathrm{Aut}(R,\Delta)$ is the abstract group of automorphisms of some based root datum of (R,Δ) of G_0 (cf. [Co14, Thm. 7.1.9]). Since A'/A is a finite Galois cover with group $\Gamma:=\mathrm{Aut}(\kappa'/\kappa)$, we have

$$H^1\big(A'/A,\;\mathrm{Aut}(R,\Delta)_A\big)\;=\;\mathrm{Hom}_{\mathrm{Grps}}(\Gamma,\;\mathrm{Aut}(R,\Delta))\;=\;H^1\big(\kappa'/\kappa,\;\mathrm{Aut}(R,\Delta)_\kappa\big),$$

i.e., the quasi-split form G^* of G is defined over κ. Now the isomorphism class of G corresponds to a class in the cohomology set

$$H^1\big(A'/A,\ G^*_{\mathrm{ad}} \otimes_\kappa A\big).$$

We claim that for any smooth affine κ-group H, the natural reduction map

$$H^1\big(A'/A,\ H \otimes_\kappa A\big) \ \to\ H^1\big(\kappa'/\kappa,\ H\big) \tag{0.3}$$

is bijective. This claim applied to $H = G^*_{\mathrm{ad}}$ implies the lemma. To prove the claim we observe that the map is surjective because any torsor on $\mathrm{Spec}(\kappa)$ can be extended constantly to a torsor on $\mathrm{Spec}(A)$ using the section $\kappa \to A$. To show that (0.3) is injective, it suffices (by the twisting trick) to prove the triviality of the kernel. Let $\mathcal{E} \to \mathrm{Spec}(A)$ be an H-torsor, and suppose $\mathcal{E}|_\kappa$ is trivial, equivalently $\mathcal{E}(\kappa)$ is non-empty. As H is a smooth, affine κ-group scheme, the torsor $\mathcal{E} \to \mathrm{Spec}(A)$ is representable by a smooth affine scheme which follows from effectivity of descent for affine maps, cf. [StaPro, 0246, 02VL]. Hence, the reduction map $\mathcal{E}(A) \to \mathcal{E}(\kappa)$ is surjective because A is Henselian, cf. [EGA IV$_4$, Thm. 18.5.17]. Thus, $\mathcal{E}(A)$ is non-empty, or equivalently $\mathcal{E}$ is trivial. This proves the claim. □

On the positive side, we have the following result. Let G be a smooth, affine F-group scheme. We denote by $\mathrm{Gr}_G := \mathrm{Gr}(\mathcal{G}, X)_\eta$ the generic fibre where $\mathcal{G} \to X$ denotes a smooth, affine model of G (this exists after possibly shrinking X). Then Gr_G is well-defined up to isomorphism independently of the choice of the model $(\mathcal{G}, X)$, and representable by a separated F-ind-scheme of ind-finite type.

Recall from [CGP10, Def. B.2.1] that a smooth, connected, unipotent F-group U is called F-wound if every map of F-schemes $\mathbb{A}^1_F \to U$ is a constant map to a point in $U(F)$.

Lemma 0.4. *Let G be a smooth, affine F-group. If $\mathrm{Gr}_G \to \mathrm{Spec}(F)$ is ind-proper, then the neutral component G^o is quasi-reductive in the sense of* [BT84, 1.1.12], *i.e., the unipotent radical $U := R_u(G^o)$ is F-wound.*

Remark 0.5. Lemma 0.4 shows that for characteristic zero fields F the unipotent radical $R_u(G^o)$ is trivial, so that G^o is indeed reductive. If F is of positive characteristic (and hence non-perfect), then by Example 0.6 below ind-properness of $\mathrm{Gr}_G \to \mathrm{Spec}(F)$ does not imply that G^o is pseudo-reductive in the sense of [CGP10, Def. 1.1.1], i.e., the unipotent radical U needs not to be trivial. It would be interesting to see whether the converse of Lemma 0.4 holds, i.e., whether quasi-reductive F-groups G are characterized by the property that $\mathrm{Gr}_G \to \mathrm{Spec}(F)$ is ind-proper.

Proof. We use the following principle: If $H \subset G$ is an F-smooth, closed, normal subgroup, and if $\mathrm{Gr}_G \to \mathrm{Spec}(F)$ is ind-proper, then $\mathrm{Gr}_H \to \mathrm{Spec}(F)$ is ind-proper as well. Indeed, the fppf (or étale) quotient G/H is an affine scheme[1], and therefore $\mathrm{Gr}_H \to \mathrm{Gr}_G$ is representable by a closed immersion, cf. [HR, Prop. 3.9]. The principle shows that if $\mathrm{Gr}_G \to \mathrm{Spec}(F)$ is ind-proper, then $\mathrm{Gr}_U \to \mathrm{Spec}(F)$ is ind-proper as well. Now by Tits' structure theory for unipotent groups [CGP10, App. B] there exists a short exact sequence $1 \to U_s \to U \to U/U_s \to 1$ where U_s is an F-split, connected, unipotent, normal subgroup of U, and the quotient U/U_s is an F-wound, unipotent group, cf. [CGP10, Thm. B.3.4]. Note that $\mathrm{Gr}_{\mathbb{G}_{a,F}} \simeq \mathrm{colim}_{n\geq 0}\, \mathbb{A}^n_F$ is an infinite-dimensional affine space, so in particular not ind-proper. Hence, if $\mathrm{Gr}_U \to \mathrm{Spec}(F)$ is ind-proper, then our principle implies that U_s is trivial, so that U is F-wound. □

Example 0.6. The following example is due to J. Lourenço. Let $F = \mathbb{F}_p(t)$ and consider the inseparable field extension $\tilde{F} = \mathbb{F}_p(t^{\frac{1}{p}})$. Define the smooth, affine, connected F-group G by the exact sequence

$$1 \to \mathbb{G}_{m,F} \to \mathrm{Res}_{\tilde{F}/F}(\mathbb{G}_{m,\tilde{F}}) \to G \to 1.$$

[1] Infact, G/H is representable by a smooth, affine F-group scheme, cf. [Bo69, §II, Thm. 6.8] (see [CGP10, Def. A.1.11] for a discussion of the different notion of quotients).

Then G is commutative, unipotent of dimension $p-1>0$ (cf. [CGP10, Exam. 1.1.3]), and therefore not pseudo-reductive. However, since $H^1_{\text{ét}}(K(\!(z)\!), \mathbb{G}_m)$ vanishes for any field extension K/F, it is easy to see that the map of ind-schemes

$$\mathrm{Gr}_{\mathrm{Res}_{F'/F}(\mathbb{G}_{m,F})} \to \mathrm{Gr}_G$$

is surjective. As the source of this map is ind-proper over F and the target is separated, we see that $\mathrm{Gr}_G \to \mathrm{Spec}(F)$ is ind-proper as well.

0.3. **On Definition 3.3.** In this definition, the Beilinson-Drinfeld affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ is defined for any smooth, affine group scheme $\mathcal{G}$ over $k[\![t]\!]$ with (connected) reductive generic fibre. The definition uses a spreading $\mathcal{G}_X$ of $\mathcal{G}$ over some smooth, affine, pointed k-curve (X, x), i.e., one has $\hat{\mathcal{O}}_{X,x} \simeq k[\![t]\!]$ on completed local rings, and $\mathcal{G}_X \to X$ is a smooth, X-affine group scheme with $\mathcal{G}_X \otimes_X \hat{\mathcal{O}}_{X,x} \simeq \mathcal{G}$. A direct way of defining $\mathrm{Gr}_{\mathcal{G}} \to \mathrm{Spec}(k[\![t]\!])$ without using a spreading $\mathcal{G}_X$ is as follows.

We define $\mathrm{Gr}_{\mathcal{G}}$ to be the functor on the category of $k[\![t]\!]$-algebras R given by the isomorphism classes of tuples $(\mathcal{E}, \alpha)$ with

$$\begin{cases} \mathcal{E} \text{ a } \mathcal{G} \otimes_{k[\![t]\!]} R[\![z-t]\!]\text{-torsor on } \mathrm{Spec}(R[\![z-t]\!]), \\ \alpha\colon \mathcal{E}|_{\mathrm{Spec}(R(\!(z-t)\!))} \simeq \mathcal{E}_0|_{\mathrm{Spec}(R(\!(z-t)\!))} \text{ a trivialization,} \end{cases} \tag{0.4}$$

where $\mathcal{E}_0$ denotes the trivial torsor. Here z is an additional formal variable, and the map $k[\![t]\!] \to R[\![z-t]\!]$ is the unique k-algebra map with the property $t \mapsto z$: the existence of this map is verified by writing a power series in z as

$$\sum_{i\geq 0} a_i z^i \;=\; \sum_{i \geq 0} b_i (z-t)^i \;\in\; R[\![z-t]\!],$$

where $a_i \in k$ are some coefficients, and $b_i = b_i(\{a_i\}, t) \in R$ are power series in t determined by the equation. The functor (0.4) agrees with Definition 3.3 defined using a spreading of $\mathcal{G}$ over some curve, and is therefore representable by an ind-projective ind-scheme over $k[\![t]\!]$. The generic fibre of $\mathrm{Gr}_{\mathcal{G}}$ is canonically the affine Grassmannian for the reductive group scheme $\mathcal{G} \otimes_{k[\![t]\!]} F[\![z-t]\!] \simeq G \otimes_F F[\![z-t]\!]$ (cf. Lemma 0.2) where $F := k(\!(t)\!)$ and $G := \mathcal{G} \otimes_{k[\![t]\!]} F$. Its special fibre is canonically the twisted affine flag variety for $\mathcal{G}$ over $k[\![t]\!]$ in the sense of [PR08].

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