SEMINAR ON THE UNRAMIFIED DRINFELD-LANGLANDS CORRESPONDENCE

Proposal for program: Timo Richarz

In the seminar we study some mathematical highlights emerging from the geometric Langlands program.

Context.[1] The global Langlands correspondence for $G = \mathrm{GL}(n)$ over a finite extension of $\mathbb{F}_p(t)$ was proven by Drinfeld [Dr1] ($n = 2$) and Lafforgue [Laf] (n arbitrary). The Drinfeld-Langlands correspondence, also called the geometric Langlands correspondence, is a conjecture analogous to the Langlands correspondence for a reductive group over a finite extension F of $k(t)$, where k is an arbitrary field. If X is a quasi-projective smooth curve over k with function field F, then the correspondence states a conjectural duality between the moduli space of G-bundles over X and the moduli space of G^L-local systems on X. For $G = \mathrm{GL}(1)$ the Drinfeld-Langlands correspondence is the geometric class field theory of Rosenlicht and Lang (cf. Serre [Se]). The case $G = \mathrm{GL}(n)$ was treated by Drinfeld in [Dr2] and [Dr3] ($n = 2$) and by Frenkel, Gaitsgory and Vilonen [FGV] (n arbitrary) for unramified local systems relying on a construction of Laumon [Lau2].

1. Overview

Follow Laumon's exposition in [Lau1], and formulate the statement of a Drinfeld-Langlands correspondence roughly. Explain the statement and construction of Beilinson-Drinfeld's correspondence in [BD1].

2. Geometric class field theory

Follow Deligne's proof of geometric class field theory [De, §e], and derive the following statement from [*loc. cit.*]: Pulling back local systems along the morphism $\pi : X \to J(X)$ induces a bijection between 1-dimensional Weil-$\overline{\mathbb{Q}}_l$-local sytems (rigidified in 0) on $J(X)$ such that

$$m^*\mathcal{L} \cong {\mathrm{pr}_1}^*\mathcal{L} \otimes {\mathrm{pr}_2}^*\mathcal{L},$$

where m denotes the group law on $J(X)$, and 1-dimensional Weil-$\overline{\mathbb{Q}}_l$-local sytems on X. Denote by $F = \mathcal{O}_{X,\eta}$ the function field of X. Applying the Grothendieck dictionary and varying the ramification gives a bijection

$$\mathrm{Hom}_{\mathrm{cont}}(F^\times \backslash \mathbb{A}_F^\times, \overline{\mathbb{Q}}_l^\times) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{cont}}(W_F, \overline{\mathbb{Q}}_l^\times),$$

which is compatible with Frobenius almost everywhere, hence is global class field theory. See Laumon [Lau3, §?] for local systems on commutative group schemes, and also his exposition [Lau1, §1] in the complex case.

[1] See Laumon's exposition in [Lau1].

3. Geometry of Bun_G and Hecke eigensheaves

The G-bundles Bun_G on X form a smooth algebraic stack locally of finite type (cf. [Hein, Prop. 1] and [LS, Prop. 3.4]). Infact, Bun_G is equidimensional of pure dimension $(g-1)\dim(G)$ (for $g > 1$), and the group of connected components $\pi_0(\mathrm{Bun}_G)$ can be canonically identified with $\pi_1(G)$. Introduce the loop groups LG, L^+G, and explain the uniformization of Bun_G [LS, Thm. 1.3]. Give a reminder on perverse sheaves, and explain the faisceaux-fonctions dictionary of Grothendieck [Lau3, §1.1]. In particular, mention Theorem 1.1 of [*loc. cit.*]. Finally introduce the notion of Hecke eigensheaves [Lau1, Def. 2.1], and state Theorem 2.2 [*loc. cit.*].

4. The unramified global Drinfeld-Langlands correspondence for $\mathrm{GL}(n)$

Explain the construction of the perverse sheaf Aut'_E in [Lau1, §4]. Make the connection to the case of GL(1) [*loc. cit.,* §1]. Follow [Lau1, §5-7] and explain why the perverse sheaf Aut'_E descends to a perverse sheaf Aut_E on Bun_G with the Hecke eigenproperty. The constructions from [*loc. cit.,* §0] are needed.

5. Geometric Satake Isomorphism I

Introduce the affine Grassmannian Gr, this is an ind-projective ind-scheme. Then explain its stratification in $G_{\mathcal{O}}$-orbits Gr^λ indexed by dominant coweights λ (Cartan decomposition), and its stratification in U-orbits S_ν indexed by all coweights ν (Iwasawa decomposition). State the Theorem of Malkin, Ostrik and Vybornov [MOV] (The closure $\overline{\mathrm{Gr}}^\lambda$ of Gr^λ is singular along its boundary.) without proof. Prove Theorem 3.2 of [MV] on the dimension of the intersection $S_\nu \cap \overline{\mathrm{Gr}}^\lambda$, and deduce Cor. 3.4 [*loc. cit.*]. Then show that the convolution product is a stratified semismall map [*loc. cit.,* Lemma 4.4.]. Mention the result of Haines [Ha, Thm. 1.1] that the fiberdimension of the convolution morphism is the maximal possibly allowed by the semismallness, if all coweights are minuscule. If some time is left, then explain that this implies (using the criterion of van der Waerden) the Theorem of [MOV] in the case of $\mathrm{GL}(n)$.

6. Geometric Satake Isomorphism II

Define the category of perverse sheaves on an ind-scheme [Na, 2.2]. Introduce the category $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr})$ of $G_{\mathcal{O}}$-equivariant sheaves on Gr. Mention Proposition 3.2.2 from [Na]. Define the convolution of perverse sheaves, and deduce from the semismallness of the convolution morphism that $\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr})$ is stable under convolution [MV, §4]. Then introduce the weight functors and prove Theorem 3.5 and 3.6 from [*loc. cit.*], and deduce Corollary 3.7 [*loc. cit.*]. If time permits, mention Lemma 3.9 and Proposition 3.10 [*loc. cit.*]. Then show that the convolution of two perverse sheaves is commutative [*loc. cit.,* §5], and sketch that H^* is a tensor functor [*loc. cit.,* §6].

7. Geometric Satake Isomorphism III

State the Main Theorem 12.1 [MV], i.e. there is an equivalence of Tannaka categories

$$\mathrm{P}_{G_{\mathcal{O}}}(\mathrm{Gr}) \xrightarrow{\cong} \mathrm{Rep}(\hat{G}),$$

where $\hat{G}$ denotes the Langlands dual group of G. Then explain the proof as presented in [*loc. cit.,* §11-12]. Use the results from [*loc. cit.,* §9-10] without proof. See also [Na, §11].

8. Center of the Iwahori-Hecke algebra

Explain the classical result which identifies the center of the Iwahori-Hecke algebra with the spherical Hecke algebra [Lu]. Then explain the construction of the geometric version due to Gaitsgory [Ga, Thm. 1] by using a variant of the Beilinson-Drinfeld Grassmannian.

References

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